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On Sum Formulas for Generalized Tribonacci Sequence

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the sum formulas for generalized Tribonacci numbers are presented. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Narayana and some other third-order linear recurrance sequences.

Keywords: Tribonacci numbers; Padovan numbers; Perrin numbers; sum formulas.

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1 INTRODUCTION

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \;\; n \geq 3 \tag{1.1}$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers. The generalized Tribonacci sequence has been studied by many authors, see for example [1-14].

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The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values.

Table 1. A few special case of generalized Tribonacci sequences

No	Sequences (Numbers)	Notation	OEIS [15]	References
1	Tribonacci	$\{T_n\} = \{W_n(0,1,1;1,1,1)\}$	A000073, A057597	[16]
2	Tribonacci-Lucas	$\{K_n\} = \{W_n(3,1,3;1,1,1)\}$	A001644, A073145	[16]
3	Tribonacci-Perrin	$\{M_n\} = \{W_n(3,0,2;1,1,1)\}$		[16]
4	modified Tribonacci	$\{U_n\} = \{W_n(1,1,1;1,1,1)\}$		[16]
5	modified Tribonacci-Lucas	$\{G_n\} = \{W_n(4,4,10;1,1,1)\}$		[16]
6	adjusted Tribonacci-Lucas	$\{H_n\} = \{W_n(4,2,0;1,1,1)\}$		[16]
7	third order Pell	$\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$	A077939, A077978	[17]
8	third order Pell-Lucas	$\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$	A276225, A276228	[17]
9	third order modified Pell	$\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$	A077997, A078049	[17]
10	third order Pell-Perrin	$\{R_n^{(3)}\} = \{W_n(3,0,2;2,1,1)\}$		
11	Padovan (Cordonnier)	$\{P_n\} = \{W_n(1,1,1;0,1,1)\}$	A000931	[18]
12	Perrin (Padovan-Lucas)	$\{E_n\} = \{W_n(3,0,2;0,1,1)\}$	A001608, A078712	[18]
13	Padovan-Perrin	$\{S_n\} = \{W_n(0,0,1;0,1,1)\}$	A000931, A176971	[18]
14	modified Padovan	$\{A_n\} = \{W_n(3,1,3;0,1,1)\}$		[18]
15	Pell-Padovan	$\{R_n\} = \{W_n(1,1,1;0,2,1)\}$	A066983, A128587	[19]
16	Pell-Perrin	$\{C_n\} = \{W_n(3,0,2;0,2,1)\}$	-	[19]
17	third order Fibonacci-Pell	$\{G_n\} = \{W_n(1,0,2;0,2,1)\}$		[19]
18	third order Lucas-Pell	$\{B_n\} = \{W_n(3,0,4;0,2,1)\}$		[19]
19	Jacobsthal-Padovan	$\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$	A159284	[20]
20	Jacobsthal-Perrin (-Lucas)	$\{L_n\} = \{W_n(3,0,2;0,1,2)\}$	A072328	[20]
21	adjusted Jacobsthal-Padovan	$\{K_n\} = \{W_n(0,1,0;0,1,2)\}$		[20]
22	modified Jacobsthal-Padovan	$\{M_n\} = \{W_n(3,1,3;0,1,2)\}$		[20]
23	Narayana	$\{N_n\} = \{W_n(0,1,1;1,0,1)\}$	A078012	[21]
24	Narayana-Lucas	$\{U_n\} = \{W_n(3,1,1;1,0,1)\}$	A001609	[21]
25	Narayana-Perrin	$\{H_n\} = \{W_n(3,0,2;1,0,1)\}$		[21]
26	third order Jacobsthal	$\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$	A077947	[22]
27	third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$	A226308	[22]
28	modified third order Jacobsthal-Lucas	$\{K_n^{(3)}\} = \{W_n(3,1,3;1,1,2)\}$		[22]
29	third order Jacobsthal-Perrin	$\{Q_n^{(3)}\} = \{W_n(3,0,2;1,1,2)\}$		[22]
30	3-primes	$\{G_n\} = \{W_n(0,1,2;2,3,5)\}$		[23]
31	Lucas 3-primes	$\{H_n\} = \{W_n(3,2,10;2,3,5)\}$		[23]
32	modified 3-primes	$\{E_n\} = \{W_n(0,1,1;2,3,5)\}$		[23]
33	reverse 3-primes	$\{N_n\} = \{W_n(0,1,5;5,3,2)\}$		[24]
34	reverse Lucas 3-primes	$\{S_n\} = \{W_n(3,5,31;5,3,2)\}$		[24]
35	reverse modified 3-primes	$\{U_n\} = \{W_n(0,1,4;5,3,2)\}$		[24]

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The evaluation of sums of these sequences is a challenging issue. Two interesting examples are

$$\sum_{k=0}^{n} kT_k = \frac{1}{2}(nT_{n+3} - T_{n+2} - (n+1)\,T_{n+1} + 2)$$

and

$$\sum_{k=1}^{n} kT_{-k} = \frac{1}{2}(-3\,(n+2)\,T_{-n-1} - (2n+5)\,T_{-n-2} - (n+3)\,T_{-n-3} + 2).$$

In this work, we derive expressions for sums of generalized Tribonacci numbers. We present some studies on summing formulas of the numbers in the following Table 2.

Table 2. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[25],[26,27]
Generalized Fibonacci	[28,29,30,31,32]
Generalized Tribonacci	[33,34,35]
Generalized Tetranacci	[36,37,38]
Generalized Pentanacci	[39,40]
Generalized Hexanacci	[41]

The following Theorem presents some sum formulas of generalized Tribonacci numbers with positive subscripts.

Theorem 1.1. *For $n \geq 0,$ we have the following formulas:*

(a) *If $r+s+t-1 \neq 0,$ then*

$$\sum_{k=0}^{n} W_k = \frac{\Omega_1}{r+s+t-1}$$

where

$$\Omega_1 = W_{n+3} + (1-r)W_{n+2} + (1-r-s)W_{n+1} - W_2 + (r-1)W_1 + (r+s-1)W_0.$$

(b) *If $(r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k} = \frac{\Omega_2}{(r+s+t-1)(r-s+t+1)}$$

where

$$\Omega_2 = (-s+1)W_{2n+2} + (t+rs)W_{2n+1} + (t^2+rt)W_{2n} + (-1+s)W_2 + (-t-rs)W_1 + (-1+r^2-s^2+rt+2s)W_0.$$

(c) *If $(r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=0}^{n} W_{2k+1} = \frac{\Omega_3}{(r-s+t+1)(r+s+t-1)}$$

where

$$\Omega_3 = (r+t)W_{2n+2} + (s-s^2+t^2+rt)W_{2n+1} + (t-st)W_{2n} + (-r-t)W_2 + (-1+s+r^2+rt)W_1 + (-t+st)W_0.$$

Proof. It is given in [35].

The following theorem presents some sum formulas (identities) of generalized Tribonacci numbers with negative subscripts.

Theorem 1.2. *For $n \geq 1,$ we have the following formulas:*

(a) *If $r+s+t-1 \neq 0$ then*

$$\sum_{k=1}^{n} W_{-k} = \frac{\Omega_4}{r+s+t-1}$$

where

$$\Omega_4 = -(r+s+t)W_{-n-1} - (s+t)W_{-n-2} - tW_{-n-3} + W_2 + (1-r)W_1 + (1-r-s)W_0.$$

(b) *If* $(r+s+t-1)(r-s+t+1) \neq 0$ *then*

$$\sum_{k=1}^{n} W_{-2k} = \frac{\Omega_5}{(r+s+t-1)(r-s+t+1)}$$

where

$$\Omega_5 = -(r+t)W_{-2n+1} + (r^2+rt+s-1)W_{-2n} + (st-t)W_{-2n-1} + (1-s)W_2 + (t+rs)W_1 + (1-rt-2s-r^2+s^2)W_0.$$

(c) *If* $(r+s+t-1)(r-s+t+1) \neq 0$ *then*

$$\sum_{k=1}^{n} W_{-2k+1} = \frac{\Omega_6}{(r+s+t-1)(r-s+t+1)}$$

where

$$\Omega_6 = (s-1)W_{-2n+1} - (t+rs)W_{-2n} - (t^2+rt)W_{-2n-1} + (r+t)W_2 + (1-r^2-rt-s)W_1 + (t-st)W_0.$$

Proof. It is given in [35].

2 SUM FORMULAS OF GENERALIZED TRIBONACCI NUMBERS WITH POSITIVE SUBSCRIPTS

The following Theorem presents some linear summing formulas of generalized Tribonacci numbers with positive subscripts.

Theorem 2.1. *For* $n \geq 0$, *we have the following formulas:*

(a) *If* $r+s+t-1 \neq 0$ *then*

$$\sum_{k=0}^{n} kW_k = \frac{\Delta_1}{(r+s+t-1)^2},$$

(b) *if* $(r+s+t-1)(r-s+t+1) \neq 0$ *then*

$$\sum_{k=0}^{n} kW_{2k} = \frac{\Delta_2}{(r-s+t+1)^2(r+s+t-1)^2},$$

(c) *if* $(r+s+t-1)(r-s+t+1) \neq 0$ *then*

$$\sum_{k=0}^{n} kW_{2k+1} = \frac{\Delta_3}{(r-s+t+1)^2(r+s+t-1)^2},$$

where

$$\Delta_1 = \sum_{k=1}^{6} \Gamma_k,\ \Delta_2 = \sum_{k=1}^{6} \Theta_k,\ \Delta_3 = \sum_{k=1}^{6} \Phi_k,$$

with

$\Gamma_1 = (n(r+s+t-1) + 2r + s - 3)W_{n+3},$

$\Gamma_2 = -(n(r-1)(r+s+t-1) + 2r^2 + rs - 4r + t + 2)W_{n+2},$

$\Gamma_3 = -(n(r+s-1)(r+s+t-1) + r^2 + s^2 + 2rs - rt - 2r - 2s + 2t + 1)W_{n+1},$

$\Gamma_4 = -(r-t-2)W_2,$

$\Gamma_5 = (r^2 - rt - 2r + s + 2t + 1)W_1,$

$\Gamma_6 = -t(2r+s-3)W_0,$
$\Theta_1 = -(n(s-1)(r+s+t-1)(r-s+t+1)+r^2s-st^2+s^2+2t^2+2rt-2s+1)W_{2n+2},$
$\Theta_2 = (t+rs)(n(r+s+t-1)(r-s+t+1)+r^2-t^2+2s-2)W_{2n+1},$
$\Theta_3 = t(n(r+t)(r+s+t-1)(r-s+t+1)+r^3+2r^2t+rt^2-s^2t+2rs+4st-2r-3t)W_{2n},$
$\Theta_4 = (r^2s-st^2+s^2+2t^2+2rt-2s+1)W_2,$
$\Theta_5 = -(t+rs)(r^2-t^2+2s-2)W_1,$
$\Theta_6 = -t(r^3+2r^2t-s^2t+rt^2-2r-3t+2rs+4st)W_0,$
$\Phi_1 = (n(r+t)(r-s+t+1)(r+s+t-1)-t^3-r^2t+rs^2-2rt^2+2st-r-2t)W_{2n+2},$
$\Phi_2 = (n(r+s+t-1)(r-s+t+1)(s+rt-s^2+t^2)-r^2s^2+2r^2t^2+r^3t+rt^3-s^3+2st^2-3t^2+2s^2-2rt-s)W_{2n+1},$
$\Phi_3 = -t(n(s-1)(r-s+t+1)(r+s+t-1)+r^2s-st^2+2t^2+s^2+2rt-2s+1)W_{2n},$
$\Phi_4 = (t^3+r^2t-rs^2+2rt^2+2t-2st+r)W_2,$
$\Phi_5 = -(-r^2s^2+2r^2t^2+r^3t+rt^3-s^3+2st^2+2s^2-3t^2-2rt-s)W_1,$
$\Phi_6 = t(r^2s-st^2+s^2+2t^2+2rt-2s+1)W_0.$

Proof.

(a) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$tW_{n-3} = W_n - rW_{n-1} - sW_{n-2}$$

we obtain

$$\begin{array}{rcl} tnW_n &=& nW_{n+3}-rnW_{n+2}-snW_{n+1} \\ t(n-1)W_{n-1} &=& (n-1)W_{n+2}-r(n-1)W_{n+1}-s(n-1)W_n \\ t(n-2)W_{n-2} &=& (n-2)W_{n+1}-r(n-2)W_n-s(n-2)W_{n-1} \\ &\vdots& \\ t\times 3\times W_3 &=& 3\times W_6 - r\times 3\times W_5 - s\times 3\times W_4 \\ t\times 2\times W_2 &=& 2\times W_5 - r\times 2\times W_4 - s\times 2\times W_3 \\ t\times 1\times W_1 &=& 1\times W_4 - r\times 1\times W_3 - s\times 1\times W_2. \end{array}$$

If we add the equations side by side, we get

$$\begin{array}{rcl} (r+s+t-1)\sum_{k=0}^{n} kW_k &=& nW_{n+3}+(n-nr-1)W_{n+2}+(n+r-nr-ns-2)W_{n+1} \\ && +W_2+(2-r)W_1+(3-2r-s)W_0+(2r+s-3)\sum_{k=0}^{n} W_k. \end{array} \tag{2.1}$$

Then, using Theorem 1.1 (a) and solving (2.1), the required result of (a) follows.

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1}+sW_{n-2}+tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we obtain

$$\begin{array}{rcl} r(n+1)W_{2n+3} & = & (n+1)W_{2n+4}-s(n+1)W_{2n+2}-t(n+1)W_{2n+1} \\ rnW_{2n+1} & = & nW_{2n+2}-snW_{2n}-tnW_{2n-1} \\ r(n-1)W_{2n-1} & = & (n-1)W_{2n}-s(n-1)W_{2n-2}-t(n-1)W_{2n-3} \\ & \vdots & \\ r\times 4\times W_9 & = & 4\times W_{10}-s\times 4\times W_8-t\times 4\times W_7 \\ r\times 3\times W_7 & = & 3\times W_8-s\times 3\times W_6-t\times 3\times W_5 \\ r\times 2\times W_5 & = & 2\times W_6-s\times 2\times W_4-t\times 2\times W_3 \\ r\times 1\times W_3 & = & 1\times W_4-s\times 1\times W_2-t\times 1\times W_1. \end{array}$$

Now, if we add the above equations side by side, we get

$$(r+t)\sum_{k=0}^{n}kW_{2k+1}=nW_{2n+2}+(n+1)tW_{2n+1}+W_0+(1-s)\sum_{k=0}^{n}kW_{2k}-\sum_{k=0}^{n}W_{2k}-t\sum_{k=0}^{n}W_{2k+1}. \tag{2.2}$$

Similarly, using the recurrence relation

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3},$$

i.e.

$$rW_{n-1}=W_n-sW_{n-2}-tW_{n-3},$$

we write the following obvious equations;

$$\begin{array}{rcl} rnW_{2n} & = & nW_{2n+1}-snW_{2n-1}-tnW_{2n-2} \\ r(n-1)W_{2n-2} & = & (n-1)W_{2n-1}-s(n-1)W_{2n-3}-t(n-1)W_{2n-4} \\ & \vdots & \\ r\times 3\times W_6 & = & 3\times W_7-s\times 3\times W_5-t\times 3\times W_4 \\ r\times 2\times W_4 & = & 2\times W_5-s\times 2\times W_3-t\times 2\times W_2 \\ r\times 1\times W_2 & = & 1\times W_3-s\times 1\times W_1-t\times 1\times W_0 \\ r\times 0\times W_0 & = & 0\times W_1-s\times 0\times W_{-1}-t\times 0\times W_{-2}. \end{array}$$

Now, if we add the above equations side by side, we obtain

$$(r+t)\sum_{k=0}^{n}kW_{2k}=(n+1)sW_{2n+1}+t(n+1)W_{2n}+(1-s)\sum_{k=0}^{n}kW_{2k+1}-t\sum_{k=0}^{n}W_{2k}-s\sum_{k=0}^{n}W_{2k+1}. \tag{2.3}$$

Then, using Theorem 1.1 (b) and (c) and solving the system (2.2)-(2.3), the required result of (b) and (c) follow.

2.1 Special Cases

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=0}^{n}kW_k$, $\sum_{k=0}^{n}kW_{2k}$ and $\sum_{k=0}^{n}kW_{2k+1}$ for the specific case of sequence $\{W_n\}$.

Taking $r=s=t=1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.1. *If $r=s=t=1$ then for $n\geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{2}(nW_{n+3} - W_{n+2} - (n+1)\,W_{n+1} + W_2 + W_1)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{4}(-W_{2n+2} + 2nW_{2n+1} + (2n+1)\,W_{2n} + W_2 - W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{4}((2n-1)\,W_{2n+2} + 2nW_{2n+1} - W_{2n} + W_2 + W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 2.2. *For $n \geq 0$, Tribonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} kT_k = \frac{1}{2}(nT_{n+3} - T_{n+2} - (n+1)\,T_{n+1} + 2)$.

(b) $\sum_{k=0}^{n} kT_{2k} = \frac{1}{4}(-T_{2n+2} + 2nT_{2n+1} + (2n+1)\,T_{2n} + 1)$.

(c) $\sum_{k=0}^{n} kT_{2k+1} = \frac{1}{4}((2n-1)\,T_{2n+2} + 2nT_{2n+1} - T_{2n} + 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 2.3. *For $n \geq 0$, Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kK_k = \frac{1}{2}(nK_{n+3} - K_{n+2} - (n+1)\,K_{n+1} + 4)$.

(b) $\sum_{k=0}^{n} kK_{2k} = \frac{1}{4}(-K_{2n+2} + 2nK_{2n+1} + (2n+1)\,K_{2n})$.

(c) $\sum_{k=0}^{n} kK_{2k+1} = \frac{1}{4}((2n-1)\,K_{2n+2} + 2nK_{2n+1} - K_{2n} + 6)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $W_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

Corollary 2.4. *For $n \geq 0$, Tribonacci-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kM_k = \frac{1}{2}(nM_{n+3} - M_{n+2} - (n+1)\,M_{n+1} + 2)$.

(b) $\sum_{k=0}^{n} kM_{2k} = \frac{1}{4}(-M_{2n+2} + 2nM_{2n+1} + (2n+1)\,M_{2n} - 1)$.

(c) $\sum_{k=0}^{n} kM_{2k+1} = \frac{1}{4}((2n-1)\,M_{2n+2} + 2nM_{2n+1} - M_{2n} + 5)$.

Taking $W_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 2.5. *For $n \geq 0$, modified Tribonacci numbers have the following properties:*

(a) $\sum_{k=0}^{n} kU_k = \frac{1}{2}(nU_{n+3} - U_{n+2} - (n+1)\,U_{n+1} + 2)$.

(b) $\sum_{k=0}^{n} kU_{2k} = \frac{1}{4}(-U_{2n+2} + 2nU_{2n+1} + (2n+1)\,U_{2n})$.

(c) $\sum_{k=0}^{n} kU_{2k+1} = \frac{1}{4}((2n-1)\,U_{2n+2} + 2nU_{2n+1} - U_{2n} + 2)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $W_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

Corollary 2.6. *For $n \geq 0$, modified Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kG_k = \frac{1}{2}(nG_{n+3} - G_{n+2} - (n+1)\,G_{n+1} + 14)$.

(b) $\sum_{k=0}^{n} kG_{2k} = \frac{1}{4}(-G_{2n+2} + 2nG_{2n+1} + (2n+1)\,G_{2n} + 6)$.

(c) $\sum_{k=0}^{n} kG_{2k+1} = \frac{1}{4}((2n-1)\,G_{2n+2} + 2nG_{2n+1} - G_{2n} + 14)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

Corollary 2.7. *For $n \geq 0$, adjusted Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kH_k = \frac{1}{2}(nH_{n+3} - H_{n+2} - (n+1) H_{n+1} + 2)$.

(b) $\sum_{k=0}^{n} kH_{2k} = \frac{1}{4}(-H_{2n+2} + 2nH_{2n+1} + (2n+1) H_{2n} - 4)$.

(c) $\sum_{k=0}^{n} kH_{2k+1} = \frac{1}{4}((2n-1) H_{2n+2} + 2nH_{2n+1} - H_{2n} + 4)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.2. *If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{9}((3n+2) W_{n+3} - (3n+5) W_{n+2} - (6n+4) W_{n+1} + W_2 + 2W_1 - 2W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{9}(-W_{2n+2} + (3n+1) W_{2n+1} + (3n+2) W_{2n} + W_2 - W_1 - 2W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{9}((3n-1) W_{2n+2} + (3n+1) W_{2n+1} - W_{2n} + W_2 - W_1 + W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

Corollary 2.8. *For $n \geq 0$, third-order Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} kP_k = \frac{1}{9}((3n+2) P_{n+3} - (3n+5) P_{n+2} - (6n+4) P_{n+1} + 4)$.

(b) $\sum_{k=0}^{n} kP_{2k} = \frac{1}{9}(-P_{2n+2} + (3n+1) P_{2n+1} + (3n+2) P_{2n} + 1)$.

(c) $\sum_{k=0}^{n} kP_{2k+1} = \frac{1}{9}((3n-1) P_{2n+2} + (3n+1) P_{2n+1} - P_{2n} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Lucas numbers.

Corollary 2.9. *For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kQ_k = \frac{1}{9}((3n+2) Q_{n+3} - (3n+5) Q_{n+2} - (6n+4) Q_{n+1} + 4)$.

(b) $\sum_{k=0}^{n} kQ_{2k} = \frac{1}{9}(-Q_{2n+2} + (3n+1) Q_{2n+1} + (3n+2) Q_{2n} - 2)$.

(c) $\sum_{k=0}^{n} kQ_{2k+1} = \frac{1}{9}((3n-1) Q_{2n+2} + (3n+1) Q_{2n+1} - Q_{2n} + 7)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 2.10. *For $n \geq 0$, third-order modified Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} kE_k = \frac{1}{9}((3n+2) E_{n+3} - (3n+5) E_{n+2} - (6n+4) E_{n+1} + 3)$.

(b) $\sum_{k=0}^{n} kE_{2k} = \frac{1}{9}(-E_{2n+2} + (3n+1) E_{2n+1} + (3n+2) E_{2n})$.

(c) $\sum_{k=0}^{n} kE_{2k+1} = \frac{1}{9}((3n-1) E_{2n+2} + (3n+1) E_{2n+1} - E_{2n})$.

Taking $W_n = R_n$ with $R_0 = 3, R_1 = 0, R_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Perrin numbers.

Corollary 2.11. *For $n \geq 0$, third-order Pell-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kR_k = \frac{1}{9}((3n+2) R_{n+3} - (3n+5) R_{n+2} - (6n+4) R_{n+1} - 4)$.

(b) $\sum_{k=0}^{n} kR_{2k} = \frac{1}{9}(-R_{2n+2} + (3n+1) R_{2n+1} + (3n+2) R_{2n} - 4)$.

(c) $\sum_{k=0}^{n} kR_{2k+1} = \frac{1}{9}((3n-1) R_{2n+2} + (3n+1) R_{2n+1} - R_{2n} + 5)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.3. *If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = (n-2)\,W_{n+3} + (n-3)\,W_{n+2} - 2W_{n+1} + 3W_2 + 4W_1 + 2W_0.$

(b) $\sum_{k=0}^{n} kW_{2k} = -W_{2n+2} + (n-1)\,W_{2n+1} + nW_{2n} + W_2 + W_1.$

(c) $\sum_{k=0}^{n} kW_{2k+1} = (n-1)\,W_{2n+1} + (n-1)\,W_{2n+2} - W_{2n} + W_2 + W_1 + W_0.$

From the last proposition, we have the following corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

Corollary 2.12. *For $n \geq 0$, Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} kP_k = (n-2)\,P_{n+3} + (n-3)\,P_{n+2} - 2P_{n+1} + 9.$

(b) $\sum_{k=0}^{n} kP_{2k} = -P_{2n+2} + (n-1)\,P_{2n+1} + nP_{2n} + 2.$

(c) $\sum_{k=0}^{n} kP_{2k+1} = (n-1)\,P_{2n+1} + (n-1)\,P_{2n+2} - P_{2n} + 3.$

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Perrin numbers.

Corollary 2.13. *For $n \geq 0$, Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kE_k = (n-2)\,E_{n+3} + (n-3)\,E_{n+2} - 2E_{n+1} + 12.$

(b) $\sum_{k=0}^{n} kE_{2k} = -E_{2n+2} + (n-1)\,E_{2n+1} + nE_{2n} + 2.$

(c) $\sum_{k=0}^{n} kE_{2k+1} = (n-1)\,E_{2n+1} + (n-1)\,E_{2n+2} - E_{2n} + 5.$

From the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

Corollary 2.14. *For $n \geq 0$, Padovan-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kS_k = (n-2)\,S_{n+3} + (n-3)\,S_{n+2} - 2S_{n+1} + 3.$

(b) $\sum_{k=0}^{n} kS_{2k} = -S_{2n+2} + (n-1)\,S_{2n+1} + nS_{2n} + 1.$

(c) $\sum_{k=0}^{n} kS_{2k+1} = (n-1)\,S_{2n+1} + (n-1)\,S_{2n+2} - S_{2n} + 1.$

Taking $W_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of modified Padovan numbers.

Corollary 2.15. *For $n \geq 0$, modified Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} kA_k = (n-2)\,A_{n+3} + (n-3)\,A_{n+2} - 2A_{n+1} + 19.$

(b) $\sum_{k=0}^{n} kA_{2k} = -A_{2n+2} + (n-1)\,A_{2n+1} + nA_{2n} + 4.$

(c) $\sum_{k=0}^{n} kA_{2k+1} = (n-1)\,A_{2n+1} + (n-1)\,A_{2n+2} - A_{2n} + 7.$

Taking $r = 0, s = 2, t = 1$ in Theorem 2.1, we obtain the following theorem.

Theorem 2.16. *If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{4}((2n-1)\,W_{n+3} + (2n-3)\,W_{n+2} - (2n+3)\,W_{n+1} + 3W_2 + 5W_1 + W_0).$

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{2}(n\,(n+3)\,W_{2n+2} - n\,(n+1)\,W_{2n+1} - (n+2)\,(n+1)\,W_{2n} + 2W_0).$

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{2}(-n\,(n+1)\,W_{2n+2} + (n^2+3n-2)W_{2n+1} + n\,(n+3)\,W_{2n} + 2W_1).$

Proof.

(a) Taking $r=0, s=2, t=1$ in Theorem 2.1 (a), we obtain

$$\sum_{k=0}^{n} kW_k = \frac{1}{4}((2n-1)W_{n+3} + (2n-3)W_{n+2} - (2n+3)W_{n+1} + 3W_2 + 5W_1 + W_0).$$

(b) It can be proved by induction.

(c) It can be proved by induction.

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 2.17. *For $n \geq 0$, Pell-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} kR_k = \frac{1}{4}((2n-1)R_{n+3} + (2n-3)R_{n+2} - (2n+3)R_{n+1} + 9)$.

(b) $\sum_{k=0}^{n} kR_{2k} = \frac{1}{2}(n(n+3)R_{2n+2} - n(n+1)R_{2n+1} - (n+2)(n+1)R_{2n} + 2)$.

(c) $\sum_{k=0}^{n} kR_{2k+1} = \frac{1}{2}(-n(n+1)R_{2n+2} + (n^2+3n-2)R_{2n+1} + n(n+3)R_{2n} + 2)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 2.18. *For $n \geq 0$, Pell-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kC_k = \frac{1}{4}((2n-1)C_{n+3} + (2n-3)C_{n+2} - (2n+3)C_{n+1} + 9)$.

(b) $\sum_{k=0}^{n} kC_{2k} = \frac{1}{2}(n(n+3)C_{2n+2} - n(n+1)C_{2n+1} - (n+2)(n+1)C_{2n} + 6)$.

(c) $\sum_{k=0}^{n} kC_{2k+1} = \frac{1}{2}(-n(n+1)C_{2n+2} + (n^2+3n-2)C_{2n+1} + n(n+3)C_{2n})$.

From the last theorem, we have the following corollary which gives sum formulas of third order Fibonacci-Pell numbers (take $W_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 2.19. *For $n \geq 0$, third order Fibonacci-Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} kG_k = \frac{1}{4}((2n-1)G_{n+3} + (2n-3)G_{n+2} - (2n+3)G_{n+1} + 7)$.

(b) $\sum_{k=0}^{n} kG_{2k} = \frac{1}{2}(n(n+3)G_{2n+2} - n(n+1)G_{2n+1} - (n+2)(n+1)G_{2n} + 2)$.

(c) $\sum_{k=0}^{n} kG_{2k+1} = \frac{1}{2}(-n(n+1)G_{2n+2} + (n^2+3n-2)G_{2n+1} + n(n+3)G_{2n})$.

Taking $W_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last theorem, we have the following corollary which presents sum formulas of third order Lucas-Pell numbers.

Corollary 2.20. *For $n \geq 0$, third order Lucas-Pell numbers have the following properties:*

(a) $\sum_{k=0}^{n} kB_k = \frac{1}{4}((2n-1)B_{n+3} + (2n-3)B_{n+2} - (2n+3)B_{n+1} + 15)$.

(b) $\sum_{k=0}^{n} kB_{2k} = \frac{1}{2}(n(n+3)B_{2n+2} - n(n+1)B_{2n+1} - (n+2)(n+1)B_{2n} + 6)$.

(c) $\sum_{k=0}^{n} kB_{2k+1} = \frac{1}{2}(-n(n+1)B_{2n+2} + (n^2+3n-2)B_{2n+1} + n(n+3)B_{2n})$.

Taking $r=0, s=1, t=2$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.4. *If $r=0, s=1, t=2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{2}((n-1)W_{n+3} + (n-2)W_{n+2} - 2W_{n+1} + 2W_2 + 3W_1 + 2W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{4}(-W_{2n+2} + 2(n-1)W_{2n+1} + 4nW_{2n1} + W_2 + 2W_1)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{4}(2(n-1)W_{2n+2} + (4n-1)W_{2n+1} - 2W_{2n} + 2W_2 + W_1 + 2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 2.21. *For $n \geq 0$, Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} kQ_k = \frac{1}{2}((n-1)\,Q_{n+3} + (n-2)\,Q_{n+2} - 2Q_{n+1} + 7)$.

(b) $\sum_{k=0}^{n} kQ_{2k} = \frac{1}{4}(-Q_{2n+2} + 2\,(n-1)\,Q_{2n+1} + 4nQ_{2n1} + 3)$.

(c) $\sum_{k=0}^{n} kQ_{2k+1} = \frac{1}{4}(2\,(n-1)\,Q_{2n+2} + (4n-1)\,Q_{2n+1} - 2Q_{2n} + 5)$.

Taking $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Perrin numbers.

Corollary 2.22. *For $n \geq 0$, Jacobsthal-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kL_k = \frac{1}{2}((n-1)\,L_{n+3} + (n-2)\,L_{n+2} - 2L_{n+1} + 10)$.

(b) $\sum_{k=0}^{n} kL_{2k} = \frac{1}{4}(-L_{2n+2} + 2\,(n-1)\,L_{2n+1} + 4nL_{2n1} + 2)$.

(c) $\sum_{k=0}^{n} kL_{2k+1} = \frac{1}{4}(2\,(n-1)\,L_{2n+2} + (4n-1)\,L_{2n+1} - 2L_{2n} + 10)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted Jacobsthal-Padovan numbers (take $W_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$).

Corollary 2.23. *For $n \geq 0$, adjusted Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} kK_k = \frac{1}{2}((n-1)\,K_{n+3} + (n-2)\,K_{n+2} - 2K_{n+1} + 3)$.

(b) $\sum_{k=0}^{n} kK_{2k} = \frac{1}{4}(-K_{2n+2} + 2\,(n-1)\,K_{2n+1} + 4nK_{2n1} + 2)$.

(c) $\sum_{k=0}^{n} kK_{2k+1} = \frac{1}{4}(2\,(n-1)\,K_{2n+2} + (4n-1)\,K_{2n+1} - 2K_{2n} + 1)$.

Taking $W_n = M_n$ with $M_0 = 3, M_1 = 1, M_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of modified Jacobsthal-Padovan numbers.

Corollary 2.24. *For $n \geq 0$, modified Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=0}^{n} kM_k = \frac{1}{2}((n-1)\,M_{n+3} + (n-2)\,M_{n+2} - 2M_{n+1} + 15)$.

(b) $\sum_{k=0}^{n} kM_{2k} = \frac{1}{4}(-M_{2n+2} + 2\,(n-1)\,M_{2n+1} + 4nM_{2n1} + 5)$.

(c) $\sum_{k=0}^{n} kM_{2k+1} = \frac{1}{4}(2\,(n-1)\,M_{2n+2} + (4n-1)\,M_{2n+1} - 2M_{2n} + 13)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.5. *If $r = 1, s = 0, t = 1$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = (n-1)\,W_{n+3} - W_{n+2} - W_{n+1} + 2W_2 + W_1 + W_0$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{9}((3n-5)\,W_{2n+2} + (3n-2)\,W_{2n+1} + (6n-1)\,W_{2n} + 5W_2 + 2W_1 + W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{9}((6n-7)\,W_{2n+2} + (6n-1)\,W_{2n+1} + (3n-5)\,W_{2n} + 7W_2 + W_1 + 5W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 2.25. *For $n \geq 0$, Narayana numbers have the following properties:*

(a) $\sum_{k=0}^{n} kN_k = ((n-1)\,N_{n+3} - N_{n+2} - N_{n+1} + 3)$.

(b) $\sum_{k=0}^{n} kN_{2k} = \frac{1}{9}((3n-5)\,N_{2n+2} + (3n-2)\,N_{2n+1} + (6n-1)\,N_{2n} + 7)$.

(c) $\sum_{k=0}^{n} kN_{2k+1} = \frac{1}{9}((6n-7)\,N_{2n+2} + (6n-1)\,N_{2n+1} + (3n-5)\,N_{2n} + 8)$.

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 2.26. *For $n \geq 0,$ Narayana-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kU_k = ((n-1)U_{n+3} - U_{n+2} - U_{n+1} + 6)$.

(b) $\sum_{k=0}^{n} kU_{2k} = \frac{1}{9}((3n-5)U_{2n+2} + (3n-2)U_{2n+1} + (6n-1)U_{2n} + 10)$.

(c) $\sum_{k=0}^{n} kU_{2k+1} = \frac{1}{9}((6n-7)U_{2n+2} + (6n-1)U_{2n+1} + (3n-5)U_{2n} + 23)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

Corollary 2.27. *For $n \geq 0,$ Narayana-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kH_k = ((n-1)H_{n+3} - H_{n+2} - H_{n+1} + 7)$.

(b) $\sum_{k=0}^{n} kH_{2k} = \frac{1}{9}((3n-5)H_{2n+2} + (3n-2)H_{2n+1} + (6n-1)H_{2n} + 13)$.

(c) $\sum_{k=0}^{n} kH_{2k+1} = \frac{1}{9}((6n-7)H_{2n+2} + (6n-1)H_{2n+1} + (3n-5)H_{2n} + 29)$.

Taking $r = 1, s = 1, t = 2$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.6. *If $r = 1, s = 1, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{3}(nW_{n+3} - W_{n+2} - (n+1)W_{n+1} + W_2 + W_1)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{9}(-W_{2n+2} + (3n-1)W_{2n+1} + 2(3n+1)W_{2n} + W_2 + W_1 - 2W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{9}((3n-2)W_{2n+2} + (6n+1)W_{2n+1} - 2W_{2n} + 2W_2 - W_1 + 2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 2.28. *For $n \geq 0,$ third order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=0}^{n} kJ_k = \frac{1}{3}(nJ_{n+3} - J_{n+2} - (n+1)J_{n+1} + J_2 + J_1)$.

(b) $\sum_{k=0}^{n} kJ_{2k} = \frac{1}{9}(-J_{2n+2} + (3n-1)J_{2n+1} + 2(3n+1)J_{2n} + J_2 + J_1 - 2J_0)$.

(c) $\sum_{k=0}^{n} kJ_{2k+1} = \frac{1}{9}((3n-2)J_{2n+2} + (6n+1)J_{2n+1} - 2J_{2n} + 2J_2 - J_1 + 2J_0)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

Corollary 2.29. *For $n \geq 0,$ third order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kj_k = \frac{1}{3}(nj_{n+3} - j_{n+2} - (n+1)j_{n+1} + j_2 + j_1)$.

(b) $\sum_{k=0}^{n} kj_{2k} = \frac{1}{9}(-j_{2n+2} + (3n-1)j_{2n+1} + 2(3n+1)j_{2n} + j_2 + j_1 - 2j_0)$.

(c) $\sum_{k=0}^{n} kj_{2k+1} = \frac{1}{9}((3n-2)j_{2n+2} + (6n+1)j_{2n+1} - 2j_{2n} + 2j_2 - j_1 + 2j_0)$.

From the last proposition, we have the following corollary which gives sum formulas of modified third order Jacobsthal-Lucas numbers (take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$).

Corollary 2.30. *For $n \geq 0,$ modified third order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=0}^{n} kK_k = \frac{1}{3}(nK_{n+3} - K_{n+2} - (n+1)K_{n+1} + K_2 + K_1)$.

(b) $\sum_{k=0}^{n} kK_{2k} = \frac{1}{9}(-K_{2n+2} + (3n-1)K_{2n+1} + 2(3n+1)K_{2n} + K_2 + K_1 - 2K_0)$.

(c) $\sum_{k=0}^{n} kK_{2k+1} = \frac{1}{9}((3n-2)K_{2n+2} + (6n+1)K_{2n+1} - 2K_{2n} + 2K_2 - K_1 + 2K_0)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Perrin numbers.

Corollary 2.31. *For $n \geq 0$, third order Jacobsthal-Perrin numbers have the following properties:*

(a) $\sum_{k=0}^{n} kQ_k = \frac{1}{3}(nQ_{n+3} - Q_{n+2} - (n+1)\,Q_{n+1} + Q_2 + Q_1)$.

(b) $\sum_{k=0}^{n} kQ_{2k} = \frac{1}{9}(-Q_{2n+2} + (3n-1)\,Q_{2n+1} + 2\,(3n+1)\,Q_{2n} + Q_2 + Q_1 - 2Q_0)$.

(c) $\sum_{k=0}^{n} kQ_{2k+1} = \frac{1}{9}((3n-2)\,Q_{2n+2} + (6n+1)\,Q_{2n+1} - 2Q_{2n} + 2Q_2 - Q_1 + 2Q_0)$.

Taking $r = 2, s = 3, t = 5$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.7. *If $r = 2, s = 3, t = 5$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{81}((9n+4)\,W_{n+3} - (9n+13)\,W_{n+2} - (36n+16)\,W_{n+1} + 5W_2 + 4W_1 - 20W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{2025}(-(90n+11)\,W_{2n+2} + 11\,(45n-17)\,W_{2n+1} + 5\,(315n+106)\,W_{2n} + 11W_2 + 187W_1 - 530W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{2025}((315n-209)\,W_{2n+2} + (1305n+497)\,W_{2n+1} - 5\,(90n+11)\,W_{2n} + 209W_2 + 55W_0 - 497W_1)$.

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 2.32. *For $n \geq 0$, 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kG_k = \frac{1}{81}((9n+4)\,G_{n+3} - (9n+13)\,G_{n+2} - (36n+16)\,G_{n+1} + 14)$.

(b) $\sum_{k=0}^{n} kG_{2k} = \frac{1}{2025}(-(90n+11)\,G_{2n+2} + 11\,(45n-17)\,G_{2n+1} + 5\,(315n+106)\,G_{2n} + 209)$.

(c) $\sum_{k=0}^{n} kG_{2k+1} = \frac{1}{2025}((315n-209)\,G_{2n+2} + (1305n+497)\,G_{2n+1} - 5\,(90n+11)\,G_{2n} - 79)$.

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 2.33. *For $n \geq 0$, Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kH_k = \frac{1}{81}((9n+4)\,H_{n+3} - (9n+13)\,H_{n+2} - (36n+16)\,H_{n+1} - 2)$.

(b) $\sum_{k=0}^{n} kH_{2k} = \frac{1}{2025}(-(90n+11)\,H_{2n+2} + 11\,(45n-17)\,H_{2n+1} + 5\,(315n+106)\,H_{2n} - 1106)$.

(c) $\sum_{k=0}^{n} kH_{2k+1} = \frac{1}{2025}((315n-209)\,H_{2n+2} + (1305n+497)\,H_{2n+1} - 5\,(90n+11)\,H_{2n} + 1261)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 2.34. *For $n \geq 0$, modified 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kE_k = \frac{1}{81}((9n+4)\,E_{n+3} - (9n+13)\,E_{n+2} - (36n+16)\,E_{n+1} + 9)$.

(b) $\sum_{k=0}^{n} kE_{2k} = \frac{1}{2025}(-(90n+11)\,E_{2n+2} + 11\,(45n-17)\,E_{2n+1} + 5\,(315n+106)\,E_{2n} + 198)$.

(c) $\sum_{k=0}^{n} kE_{2k+1} = \frac{1}{2025}((315n-209)\,E_{2n+2} + (1305n+497)\,E_{2n+1} - 5\,(90n+11)\,E_{2n} - 288)$.

Taking $r = 5, s = 3, t = 2$ in Theorem 2.1, we obtain the following proposition.

Proposition 2.8. *If $r = 5, s = 3, t = 2$ then for $n \geq 0$ we have the following formulas:*

(a) $\sum_{k=0}^{n} kW_k = \frac{1}{81}((9n+10)\,W_{n+3} - (36n+49)\,W_{n+2} - (63n+43)\,W_{n+1} - W_2 + 13W_1 - 20W_0)$.

(b) $\sum_{k=0}^{n} kW_{2k} = \frac{1}{405}(-(18n+19)\,W_{2n+2} + 17\,(9n+5)\,W_{2n+1} + 2\,(63n+53)\,W_{2n} + 19W_2 - 85W_1 - 106W_0)$.

(c) $\sum_{k=0}^{n} kW_{2k+1} = \frac{1}{405}((63n-10)W_{2n+2}+(72n+49)W_{2n+1}-2(18n+19)W_{2n}+10W_2-49W_1+38W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 2.35. *For $n \geq 0$, reverse 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kN_k = \frac{1}{81}((9n+10)N_{n+3}-(36n+49)N_{n+2}-(63n+43)N_{n+1}+8)$.

(b) $\sum_{k=0}^{n} kN_{2k} = \frac{1}{405}(-(18n+19)N_{2n+2}+17(9n+5)N_{2n+1}+2(63n+53)N_{2n}+10)$.

(c) $\sum_{k=0}^{n} kN_{2k+1} = \frac{1}{405}((63n-10)N_{2n+2}+(72n+49)N_{2n+1}-2(18n+19)N_{2n}+1)$.

Taking $W_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 2.36. *For $n \geq 0$, reverse Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kS_k = \frac{1}{81}((9n+10)S_{n+3}-(36n+49)S_{n+2}-(63n+43)S_{n+1}-26)$.

(b) $\sum_{k=0}^{n} kS_{2k} = \frac{1}{405}(-(18n+19)S_{2n+2}+17(9n+5)S_{2n+1}+2(63n+53)S_{2n}-154)$.

(c) $\sum_{k=0}^{n} kS_{2k+1} = \frac{1}{405}((63n-10)S_{2n+2}+(72n+49)S_{2n+1}-2(18n+19)S_{2n}+179)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $W_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 2.37. *For $n \geq 0$, reverse modified 3-primes numbers have the following properties:*

(a) $\sum_{k=0}^{n} kU_k = \frac{1}{81}((9n+10)U_{n+3}-(36n+49)U_{n+2}-(63n+43)U_{n+1}+9)$.

(b) $\sum_{k=0}^{n} kU_{2k} = \frac{1}{405}(-(18n+19)U_{2n+2}+17(9n+5)U_{2n+1}+2(63n+53)U_{2n}-9)$.

(c) $\sum_{k=0}^{n} kU_{2k+1} = \frac{1}{405}((63n-10)U_{2n+2}+(72n+49)U_{2n+1}-2(18n+19)U_{2n}-9)$.

3 SUM FORMULAS OF GENERALIZED TRIBONACCI NUMBERS WITH NEGATIVE SUBSCRIPTS

The following Theorem presents some sum formulas (identities) of generalized Tribonacci numbers with negative subscripts.

Theorem 3.1. *For $n \geq 1$, we have the following formulas:*

(a) *If $r+s+t-1 \neq 0$ then*

$$\sum_{k=1}^{n} kW_{-k} = \frac{\Lambda_1}{(r+s+t-1)^2},$$

(b) *if $(r+s+t-1)(r-s+t+1) \neq 0$ then*

$$\sum_{k=1}^{n} kW_{-2k} = \frac{\Lambda_2}{(r-s+t+1)^2(r+s+t-1)^2},$$

and

(c) *if* $(r+s+t-1)(r-s+t+1)\neq 0$ *then*

$$\sum_{k=1}^{n} kW_{-2k+1} = \frac{\Lambda_3}{(r-s+t+1)^2(r+s+t-1)^2},$$

where

$$\Lambda_1 = \sum_{k=1}^{6}\lambda_k,\ \Lambda_2 = \sum_{k=1}^{6}\theta_k,\ \Lambda_3 = \sum_{k=1}^{6}\mu_k,$$

with

$\lambda_1 = -(n(r+s+t)(r+s+t-1)+r^2+s^2+t^2+2rs+2rt+2st+s+2t)W_{-n-1},$
$\lambda_2 = -(n(s+t)(r+s+t-1)+2s^2+2t^2+rs+4st+t)W_{-n-2},$
$\lambda_3 = t(-n(r+s+t-1)-r-2s-3t)W_{-n-3},$
$\lambda_4 = -(r-t-2)W_2,$
$\lambda_5 = (r^2-rt-2r+s+2t+1)W_1,$
$\lambda_6 = -t(2r+s-3)W_0,$
$\theta_1 = -(n(r+t)(r-s+t+1)(r+s+t-1)+t^3-rs^2+2rt^2+r^2t-2st+r+2t)W_{-2n+1},$
$\theta_2 = (n(r-s+t+1)(r+s+t-1)(s+rt+r^2-1)+r^3t+rt^3-r^2s^2+2r^2t^2-s^3+2st^2+2s^2-3t^2-2rt-s)W_{-2n},$
$\theta_3 = t(n(s-1)(r+s+t-1)(r-s+t+1)-r^2s+st^2-s^2-2t^2-2rt+2s-1)W_{-2n-1},$
$\theta_4 = (r^2s-st^2+s^2+2t^2+2rt-2s+1)W_2,$
$\theta_5 = -(t+rs)(r^2-t^2+2s-2)W_1,$
$\theta_6 = -t(r^3+2r^2t+rt^2-s^2t+2rs+4st-2r-3t)W_0,$
$\mu_1 = (n(s-1)(r+s+t-1)(r-s+t+1)-s^3+2st^2+2rst-r^2+2s^2-3t^2-4rt-s)W_{-2n+1},$
$\mu_2 = (t+rs)(-n(r+s+t-1)(r-s+t+1)+s^2-2t^2-2rt-1)W_{-2n},$
$\mu_3 = -t(n(r+t)(r-s+t+1)(r+s+t-1)+t^3+2rt^2+r^2t-rs^2-2st+r+2t)W_{-2n-1},$
$\mu_4 = (r^2t+t^3-rs^2+2rt^2-2st+r+2t)W_2,$
$\mu_5 = -(r^3t+rt^3-r^2s^2+2r^2t^2+2st^2-s^3+2s^2-3t^2-2rt-s)W_1,$
$\mu_6 = t(r^2s-st^2+s^2+2t^2+2rt-2s+1)W_0.$

Proof.

(a) Using the recurrence relation

$$W_{-n+3} = r\times W_{-n+2}+s\times W_{-n+1}+t\times W_{-n} \Rightarrow W_{-n} = -\frac{s}{t}W_{-(n-1)}-\frac{r}{t}W_{-(n-2)}+\frac{1}{t}W_{-(n-3)}$$

i.e.

$$tW_{-n} = W_{-n+3}-rW_{-n+2}-sW_{-n+1},$$

or

$$W_{-n} = \frac{1}{t}W_{-n+3}-\frac{r}{t}W_{-n+2}-\frac{s}{t}W_{-n+1},$$

we obtain

$$\begin{aligned}
tnW_{-n} &= nW_{-n+3}-rnW_{-n+2}-snW_{-n+1}\\
t(n-1)W_{-n+1} &= (n-1)W_{-n+4}-r(n-1)W_{-n+3}-s(n-1)W_{-n+2}\\
t(n-2)W_{-n+2} &= (n-2)W_{-n+5}-r(n-2)W_{-n+4}-s(n-2)W_{-n+3}\\
&\vdots\\
t\times 3\times W_{-3} &= 3\times W_0-r\times 3\times W_{-1}-s\times 3\times W_{-2}\\
t\times 2\times W_{-2} &= 2\times W_1-r\times 2\times W_0-s\times 2\times W_{-1}\\
t\times 1\times W_{-1} &= 1\times W_2-r\times 1\times W_1-s\times 1\times W_0.
\end{aligned}$$

If we add the equations side by side, we get

$$\begin{aligned}(r+s+t-1)\sum_{k=1}^{n}kW_{-k} &= -(3r+2s+t+nr+ns+nt)W_{-n-1}-(3s+2t+ns+nt)W_{-n-2} \\ &\quad -t(n+3)W_{-n-3}+W_2+(2-r)W_1-(2r+s-3)W_0-(2r+s-3)\sum_{k=1}^{n}W_{-k}.\end{aligned} \tag{3.1}$$

Then, using Theorem 1.2 (a) and solving (3.1), the required result of (a) follows.

(b) and (c) Using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n},$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n},$$

we obtain

$$\begin{aligned} snW_{-2n+1} &= nW_{-2n+3}-rnW_{-2n+2}-tnW_{-2n} \\ s(n-1)W_{-2n+3} &= (n-1)W_{-2n+5}-r(n-1)W_{-2n+4}-t(n-1)W_{-2n+2} \\ s(n-2)W_{-2n+5} &= (n-2)W_{-2n+7}-r(n-2)W_{-2n+6}-t(n-2)W_{-2n+4} \\ &\vdots \\ s\times 3\times W_{-5} &= 3\times W_{-3}-r\times 3\times W_{-4}-t\times 3\times W_{-6} \\ s\times 2\times W_{-3} &= 2\times W_{-1}-r\times 2\times W_{-2}-t\times 2\times W_{-4} \\ s\times 1\times W_{-1} &= 1\times W_{1}-r\times 1\times W_{0}-t\times 1\times W_{-2}.\end{aligned}$$

If we add the equations side by side, we get

$$(s-1)\sum_{k=1}^{n}kW_{-2k+1}=-(n+1)W_{-2n+1}+r(n+1)W_{-2n}+W_1-rW_0-(r+t)\sum_{k=1}^{n}kW_{-2k}+\sum_{k=1}^{n}W_{-2k+1}-r\sum_{k=1}^{n}W_{-2k}. \tag{3.2}$$

Similarly, using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n}$$

we obtain

$$\begin{aligned} snW_{-2n} &= nW_{-2n+2}-rnW_{-2n+1}-tnW_{-2n-1} \\ s(n-1)W_{-2n+2} &= (n-1)W_{-2n+4}-r(n-1)W_{-2n+3}-t(n-1)W_{-2n+1} \\ s(n-2)W_{-2n+4} &= (n-2)W_{-2n+6}-r(n-2)W_{-2n+5}-t(n-2)W_{-2n+3} \\ &\vdots \\ s\times 3\times W_{-6} &= 3\times W_{-4}-r\times 3\times W_{-5}-t\times 3\times W_{-7} \\ s\times 2\times W_{-4} &= 2\times W_{-2}-r\times 2\times W_{-3}-t\times 2\times W_{-5} \\ s\times 1\times W_{-2} &= 1\times W_{0}-r\times 1\times W_{-1}-t\times 1\times W_{-3}\end{aligned}$$

If we add the equations side by side, we get

$$(s-1)\sum_{k=1}^{n}kW_{-2k}=-(n+1)W_{-2n}-tnW_{-2n-1}+W_0-(r+t)\sum_{k=1}^{n}kW_{-2k+1}+t\sum_{k=1}^{n}W_{-2k+1}+\sum_{k=1}^{n}W_{-2k}. \tag{3.3}$$

Then, using Theorem 1.2 (b) and (c) and solving system (3.2)-(3.3) the required result of (b) and (c) follow.

3.1 Special Cases

In this section, we present the closed form solutions (identities) of the sums $\sum_{k=1}^{n} kW_{-k}$, $\sum_{k=1}^{n} kW_{-2k}$ and $\sum_{k=1}^{n} kW_{-2k+1}$ for the specific case of sequence $\{W_n\}$.

Taking $r = s = t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.1. *If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{2}(-3(n+2)W_{-n-1} - (2n+5)W_{-n-2} - (n+3)W_{-n-3} + W_2 + W_1)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{4}(-(2n+1)W_{-2n+1} + 2nW_{-2n} - W_{-2n-1} + W_2 - W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{4}(-W_{-2n+1} - 2(n+1)W_{-2n} - (2n+1)W_{-2n-1} + W_2 + W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 3.2. *For $n \geq 1$, Tribonacci numbers have the following properties:*

(a) $\sum_{k=1}^{n} kT_{-k} = \frac{1}{2}(-3(n+2)T_{-n-1} - (2n+5)T_{-n-2} - (n+3)T_{-n-3} + 2)$.

(b) $\sum_{k=1}^{n} kT_{-2k} = \frac{1}{4}(-(2n+1)T_{-2n+1} + 2nT_{-2n} - T_{-2n-1} + 1)$.

(c) $\sum_{k=1}^{n} kT_{-2k+1} = \frac{1}{4}(-T_{-2n+1} - 2(n+1)T_{-2n} - (2n+1)T_{-2n-1} + 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 3.3. *For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kK_{-k} = \frac{1}{2}(-3(n+2)K_{-n-1} - (2n+5)K_{-n-2} - (n+3)K_{-n-3} + 4)$.

(b) $\sum_{k=1}^{n} kK_{-2k} = \frac{1}{4}(-(2n+1)K_{-2n+1} + 2nK_{-2n} - K_{-2n-1})$.

(c) $\sum_{k=1}^{n} kK_{-2k+1} = \frac{1}{4}(-K_{-2n+1} - 2(n+1)K_{-2n} - (2n+1)K_{-2n-1} + 6)$.

From the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Perrin numbers (take $W_n = M_n$ with $M_0 = 3, M_1 = 0, M_2 = 2$).

Corollary 3.4. *For $n \geq 1$, Tribonacci-Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kM_{-k} = \frac{1}{2}(-3(n+2)M_{-n-1} - (2n+5)M_{-n-2} - (n+3)M_{-n-3} + 2)$.

(b) $\sum_{k=1}^{n} kM_{-2k} = \frac{1}{4}(-(2n+1)M_{-2n+1} + 2nM_{-2n} - M_{-2n-1} - 1)$.

(c) $\sum_{k=1}^{n} kM_{-2k+1} = \frac{1}{4}(-M_{-2n+1} - 2(n+1)M_{-2n} - (2n+1)M_{-2n-1} + 5)$.

Taking $W_n = U_n$ with $U_0 = 1, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of modified Tribonacci numbers.

Corollary 3.5. *For $n \geq 1$, modified Tribonacci numbers have the following properties:*

(a) $\sum_{k=1}^{n} kU_{-k} = \frac{1}{2}(-3(n+2)U_{-n-1} - (2n+5)U_{-n-2} - (n+3)U_{-n-3} + 2)$.

(b) $\sum_{k=1}^{n} kU_{-2k} = \frac{1}{4}(-(2n+1)U_{-2n+1} + 2nU_{-2n} - U_{-2n-1})$.

(c) $\sum_{k=1}^{n} kU_{-2k+1} = \frac{1}{4}(-U_{-2n+1} - 2(n+1)U_{-2n} - (2n+1)U_{-2n-1} + 2)$.

From the last proposition, we have the following corollary which gives sum formulas of modified Tribonacci-Lucas numbers (take $W_n = G_n$ with $G_0 = 4, G_1 = 4, G_2 = 10$).

Corollary 3.6. *For $n \geq 1$, modified Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kG_{-k} = \frac{1}{2}(-3(n+2)G_{-n-1} - (2n+5)G_{-n-2} - (n+3)G_{-n-3} + 14)$.

(b) $\sum_{k=1}^{n} kG_{-2k} = \frac{1}{4}(-(2n+1)G_{-2n+1} + 2nG_{-2n} - G_{-2n-1} + 6)$.

(c) $\sum_{k=1}^{n} kG_{-2k+1} = \frac{1}{4}(-G_{-2n+1} - 2(n+1)G_{-2n} - (2n+1)G_{-2n-1} + 14)$.

Taking $W_n = H_n$ with $H_0 = 4, H_1 = 2, H_2 = 0$ in the last proposition, we have the following corollary which presents sum formulas of adjusted Tribonacci-Lucas numbers.

Corollary 3.7. *For $n \geq 1,$ adjusted Tribonacci-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kH_{-k} = \frac{1}{2}(-3(n+2)H_{-n-1} - (2n+5)H_{-n-2} - (n+3)H_{-n-3} + 2)$.

(b) $\sum_{k=1}^{n} kH_{-2k} = \frac{1}{4}(-(2n+1)H_{-2n+1} + 2nH_{-2n} - H_{-2n-1} - 4)$.

(c) $\sum_{k=1}^{n} kH_{-2k+1} = \frac{1}{4}(-H_{-2n+1} - 2(n+1)H_{-2n} - (2n+1)H_{-2n-1} + 4)$.

Taking $r = 2, s = 1, t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.2. *If $r = 2, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{9}(-(12n+19)W_{-n-1} - (6n+11)W_{-n-2} - (3n+7)W_{-n-3} + W_2 + 2W_1 - 2W_0)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{9}(-(3n+1)W_{-2n+1} + (6n+1)W_{-2n} - W_{-2n-1} + W_2 - W_1 - 2W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{9}(-W_{-2n+1} - (3n+2)W_{-2n} - (3n+1)W_{-2n-1} + W_2 - W_1 + W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 1$).

Corollary 3.8. *For $n \geq 1,$ third-order Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} kP_{-k} = \frac{1}{9}(-(12n+19)P_{-n-1} - (6n+11)P_{-n-2} - (3n+7)P_{-n-3} + 4)$.

(b) $\sum_{k=1}^{n} kP_{-2k} = \frac{1}{9}(-(3n+1)P_{-2n+1} + (6n+1)P_{-2n} - P_{-2n-1} + 1)$.

(c) $\sum_{k=1}^{n} kP_{-2k+1} = \frac{1}{9}(-P_{-2n+1} - (3n+2)P_{-2n} - (3n+1)P_{-2n-1} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Lucas numbers.

Corollary 3.9. *For $n \geq 1,$ third-order Pell-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kQ_{-k} = \frac{1}{9}(-(12n+19)Q_{-n-1} - (6n+11)Q_{-n-2} - (3n+7)Q_{-n-3} + 4)$.

(b) $\sum_{k=1}^{n} kQ_{-2k} = \frac{1}{9}(-(3n+1)Q_{-2n+1} + (6n+1)Q_{-2n} - Q_{-2n-1} - 2)$.

(c) $\sum_{k=1}^{n} kQ_{-2k+1} = \frac{1}{9}(-Q_{-2n+1} - (3n+2)Q_{-2n} - (3n+1)Q_{-2n-1} + 7)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 3.10. *For $n \geq 1,$ third-order modified Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} kE_{-k} = \frac{1}{9}(-(12n+19)E_{-n-1} - (6n+11)E_{-n-2} - (3n+7)E_{-n-3} + 3)$.

(b) $\sum_{k=1}^{n} kE_{-2k} = \frac{1}{9}(-(3n+1)E_{-2n+1} + (6n+1)E_{-2n} - E_{-2n-1})$.

(c) $\sum_{k=1}^{n} kE_{-2k+1} = \frac{1}{9}(-E_{-2n+1} - (3n+2)E_{-2n} - (3n+1)E_{-2n-1})$.

Taking $W_n = R_n$ with $R_0 = 3, R_1 = 0, R_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Perrin numbers.

Corollary 3.11. *For $n \geq 1,$ third-order Pell-Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kR_{-k} = \frac{1}{9}(-(12n+19)R_{-n-1} - (6n+11)R_{-n-2} - (3n+7)R_{-n-3} - 4)$.

(b) $\sum_{k=1}^{n} kR_{-2k} = \frac{1}{9}(-(3n+1)R_{-2n+1} + (6n+1)R_{-2n} - R_{-2n-1} - 4)$.

(c) $\sum_{k=1}^{n} kR_{-2k+1} = \frac{1}{9}(-R_{-2n+1} - (3n+2)R_{-2n} - (3n+1)R_{-2n-1} + 5)$.

Taking $r = 0, s = 1, t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.3. *If $r = 0, s = 1, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = -(2n+7)W_{-n-1} - (2n+9)W_{-n-2} - (n+5)W_{-n-3} + 3W_2 + 4W_1 + 2W_0$.

(b) $\sum_{k=1}^{n} kW_{-2k} = -(n+1)W_{-2n+1} - W_{-2n} - W_{-2n-1} + W_2 + W_1$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = -W_{-2n+1} - (n+2)W_{-2n} - (n+1)W_{-2n-1} + W_2 + W_1 + W_0$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P_1 = 1, P_2 = 1$).

Corollary 3.12. *For $n \geq 1$, Padovan numbers have the following properties:*

(a) $\sum_{k=1}^{n} kP_{-k} = -(2n+7)P_{-n-1} - (2n+9)P_{-n-2} - (n+5)P_{-n-3} + 9$.

(b) $\sum_{k=1}^{n} kP_{-2k} = -(n+1)P_{-2n+1} - P_{-2n} - P_{-2n-1} + 2$.

(c) $\sum_{k=1}^{n} kP_{-2k+1} = -P_{-2n+1} - (n+2)P_{-2n} - (n+1)P_{-2n-1} + 3$.

Taking $W_n = E_n$ with $E_0 = 3, E_1 = 0, E_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Perrin numbers.

Corollary 3.13. *For $n \geq 1$, Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kE_{-k} = -(2n+7)E_{-n-1} - (2n+9)E_{-n-2} - (n+5)E_{-n-3} + 12$.

(b) $\sum_{k=1}^{n} kE_{-2k} = -(n+1)E_{-2n+1} - E_{-2n} - E_{-2n-1} + 2$.

(c) $\sum_{k=1}^{n} kE_{-2k+1} = -E_{-2n+1} - (n+2)E_{-2n} - (n+1)E_{-2n-1} + 5$.

From the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers (take $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$).

Corollary 3.14. *For $n \geq 1$, Padovan-Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kS_{-k} = -(2n+7)S_{-n-1} - (2n+9)S_{-n-2} - (n+5)S_{-n-3} + 3$.

(b) $\sum_{k=1}^{n} kS_{-2k} = -(n+1)S_{-2n+1} - S_{-2n} - S_{-2n-1} + 1$.

(c) $\sum_{k=1}^{n} kS_{-2k+1} = -S_{-2n+1} - (n+2)S_{-2n} - (n+1)S_{-2n-1} + 1$.

Taking $W_n = A_n$ with $A_0 = 3, A_1 = 1, A_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of modified Padovan numbers.

Corollary 3.15. *For $n \geq 1$, modified Padovan numbers have the following properties:*

(a) $\sum_{k=1}^{n} kA_{-k} = -(2n+7)A_{-n-1} - (2n+9)A_{-n-2} - (n+5)A_{-n-3} + 19$.

(b) $\sum_{k=1}^{n} kA_{-2k} = -(n+1)A_{-2n+1} - A_{-2n} - A_{-2n-1} + 4$.

(c) $\sum_{k=1}^{n} kA_{-2k+1} = -A_{-2n+1} - (n+2)A_{-2n} - (n+1)A_{-2n-1} + 7$.

Taking $r = 0, s = 2, t = 1$ in Theorem 3.1, we obtain the following theorem.

Theorem 3.16. *If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{4}(-(6n+13)W_{-n-1} - (6n+19)W_{-n-2} - (2n+7)W_{-n-3} + 3W_2 + 5W_1 + W_0)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{2}(-n(n-1)W_{-2n+1} + (n+1)(n-2)W_{-2n} + n(n-3)W_{-2n-1} + 2W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{2}((n+1)(n-2)W_{-2n+1} - n(n+1)W_{-2n} - n(n-1)W_{-2n-1} + 2W_1)$.

Proof

(a) If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (a) then we have

$$\sum_{k=1}^{n} kW_{-k} = \frac{1}{4}(-(6n+13)W_{-n-1} - (6n+19)W_{-n-2} - (2n+7)W_{-n-3} + 3W_2 + 5W_1 + W_0).$$

(b) It can be proved by induction.

(c) It can be proved by induction.

From the last theorem, we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 3.17. *For $n \geq 1$, Pell-Padovan numbers have the following properties:*

(a) $\sum_{k=1}^{n} kR_{-k} = \frac{1}{4}(-(6n+13)R_{-n-1} - (6n+19)R_{-n-2} - (2n+7)R_{-n-3} + 9)$.

(b) $\sum_{k=1}^{n} kR_{-2k} = \frac{1}{2}(-n(n-1)R_{-2n+1} + (n+1)(n-2)R_{-2n} + n(n-3)R_{-2n-1} + 2)$.

(c) $\sum_{k=1}^{n} kR_{-2k+1} = \frac{1}{2}((n+1)(n-2)R_{-2n+1} - n(n+1)R_{-2n} - n(n-1)R_{-2n-1} + 2)$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 3.18. *For $n \geq 1$, Pell-Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kC_{-k} = \frac{1}{4}(-(6n+13)C_{-n-1} - (6n+19)C_{-n-2} - (2n+7)C_{-n-3} + 9)$.

(b) $\sum_{k=1}^{n} kC_{-2k} = \frac{1}{2}(-n(n-1)C_{-2n+1} + (n+1)(n-2)C_{-2n} + n(n-3)C_{-2n-1} + 6)$.

(c) $\sum_{k=1}^{n} kC_{-2k+1} = \frac{1}{2}((n+1)(n-2)C_{-2n+1} - n(n+1)C_{-2n} - n(n-1)C_{-2n-1})$.

From the last theorem, we have the following corollary which gives sum formulas of third order Fibonacci-Pell numbers (take $W_n = G_n$ with $G_0 = 1, G_1 = 0, G_2 = 2$).

Corollary 3.19. *For $n \geq 1$, third order Fibonacci-Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} kG_{-k} = \frac{1}{4}(-(6n+13)G_{-n-1} - (6n+19)G_{-n-2} - (2n+7)G_{-n-3} + 7)$.

(b) $\sum_{k=1}^{n} kG_{-2k} = \frac{1}{2}(-n(n-1)G_{-2n+1} + (n+1)(n-2)G_{-2n} + n(n-3)G_{-2n-1} + 2)$.

(c) $\sum_{k=1}^{n} kG_{-2k+1} = \frac{1}{2}((n+1)(n-2)G_{-2n+1} - n(n+1)G_{-2n} - n(n-1)G_{-2n-1})$.

Taking $W_n = B_n$ with $B_0 = 3, B_1 = 0, B_2 = 4$ in the last theorem, we have the following corollary which presents sum formulas of third order Lucas-Pell numbers.

Corollary 3.20. *For $n \geq 1$, third order Lucas-Pell numbers have the following properties:*

(a) $\sum_{k=1}^{n} kB_{-k} = \frac{1}{4}(-(6n+13)B_{-n-1} - (6n+19)B_{-n-2} - (2n+7)B_{-n-3} + 15)$.

(b) $\sum_{k=1}^{n} kB_{-2k} = \frac{1}{2}(-n(n-1)B_{-2n+1} + (n+1)(n-2)B_{-2n} + n(n-3)B_{-2n-1} + 6)$.

(c) $\sum_{k=1}^{n} kB_{-2k+1} = \frac{1}{2}((n+1)(n-2)B_{-2n+1} - n(n+1)B_{-2n} - n(n-1)B_{-2n-1})$.

Taking $r = 0, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.4. *If $r = 0, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{2}(-(3n+7)W_{-n-1} - (3n+10)W_{-n-2} - 2(n+4)W_{-n-3} + 2W_2 + 3W_1 + 2W_0)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{4}(-2(n+1)W_{-2n+1} - W_{-2n} - 2W_{-2n-1} + W_2 + 2W_1)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{4}(-W_{-2n+1} - 2(n+2)W_{-2n} - 4(n+1)W_{-2n-1} + 2W_2 + W_1 + 2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 3.21. *For $n \geq 1$, Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=1}^{n} kQ_{-k} = \frac{1}{2}(-(3n+7)Q_{-n-1} - (3n+10)Q_{-n-2} - 2(n+4)Q_{-n-3} + 7)$.

(b) $\sum_{k=1}^{n} kQ_{-2k} = \frac{1}{4}(-2(n+1)Q_{-2n+1} - Q_{-2n} - 2Q_{-2n-1} + 3)$.

(c) $\sum_{k=1}^{n} kQ_{-2k+1} = \frac{1}{4}(-Q_{-2n+1} - 2(n+2)Q_{-2n} - 4(n+1)Q_{-2n-1} + 5)$.

Taking $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Perrin numbers.

Corollary 3.22. *For $n \geq 1$, Jacobsthal-Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kL_{-k} = \frac{1}{2}(-(3n+7)L_{-n-1} - (3n+10)L_{-n-2} - 2(n+4)L_{-n-3} + 10)$.

(b) $\sum_{k=1}^{n} kL_{-2k} = \frac{1}{4}(-2(n+1)L_{-2n+1} - L_{-2n} - 2L_{-2n-1} + 2)$.

(c) $\sum_{k=1}^{n} kL_{-2k+1} = \frac{1}{4}(-L_{-2n+1} - 2(n+2)L_{-2n} - 4(n+1)L_{-2n-1} + 10)$.

From the last proposition, we have the following corollary which gives sum formulas of adjusted Jacobsthal-Padovan numbers (take $W_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = 0$).

Corollary 3.23. *For $n \geq 1$, adjusted Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=1}^{n} kK_{-k} = \frac{1}{2}(-(3n+7)K_{-n-1} - (3n+10)K_{-n-2} - 2(n+4)K_{-n-3} + 3)$.

(b) $\sum_{k=1}^{n} kK_{-2k} = \frac{1}{4}(-2(n+1)K_{-2n+1} - K_{-2n} - 2K_{-2n-1} + 2)$.

(c) $\sum_{k=1}^{n} kK_{-2k+1} = \frac{1}{4}(-K_{-2n+1} - 2(n+2)K_{-2n} - 4(n+1)K_{-2n-1} + 1)$.

Taking $W_n = M_n$ with $M_0 = 3, M_1 = 1, M_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of modified Jacobsthal-Padovan numbers.

Corollary 3.24. *For $n \geq 1$, modified Jacobsthal-Padovan numbers have the following properties:*

(a) $\sum_{k=1}^{n} kM_{-k} = \frac{1}{2}(-(3n+7)M_{-n-1} - (3n+10)M_{-n-2} - 2(n+4)M_{-n-3} + 15)$.

(b) $\sum_{k=1}^{n} kM_{-2k} = \frac{1}{4}(-2(n+1)M_{-2n+1} - M_{-2n} - 2M_{-2n-1} + 5)$.

(c) $\sum_{k=1}^{n} kM_{-2k+1} = \frac{1}{4}(-M_{-2n+1} - 2(n+2)M_{-2n} - 4(n+1)M_{-2n-1} + 13)$.

Taking $r = 1, s = 0, t = 1$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.5. *If $r = 1, s = 0, t = 1$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = -(2n+6)W_{-n-1} - (n+3)W_{-n-2} - (n+4)W_{-n-3} + 2W_2 + W_1 + W_0$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{9}(-(6n+7)W_{-2n+1} + (3n-1)W_{-2n} - (3n+5)W_{-2n-1} + 5W_2 + 2W_1 + W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{9}(-(3n+8)W_{-2n+1} - (3n+5)W_{-2n} - (6n+7)W_{-2n-1} + 7W_2 + W_1 + 5W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 3.25. *For $n \geq 1$, Narayana numbers have the following properties:*

(a) $\sum_{k=1}^{n} kN_{-k} = -(2n+6)N_{-n-1} - (n+3)N_{-n-2} - (n+4)N_{-n-3} + 3.$

(b) $\sum_{k=1}^{n} kN_{-2k} = \frac{1}{9}(-(6n+7)N_{-2n+1} + (3n-1)N_{-2n} - (3n+5)N_{-2n-1} + 7).$

(c) $\sum_{k=1}^{n} kN_{-2k+1} = \frac{1}{9}(-(3n+8)N_{-2n+1} - (3n+5)N_{-2n} - (6n+7)N_{-2n-1} + 8).$

Taking $W_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 3.26. *For $n \geq 1,$ Narayana-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kU_{-k} = -(2n+6)U_{-n-1} - (n+3)U_{-n-2} - (n+4)U_{-n-3} + 6.$

(b) $\sum_{k=1}^{n} kU_{-2k} = \frac{1}{9}(-(6n+7)U_{-2n+1} + (3n-1)U_{-2n} - (3n+5)U_{-2n-1} + 10).$

(c) $\sum_{k=1}^{n} kU_{-2k+1} = \frac{1}{9}(-(3n+8)U_{-2n+1} - (3n+5)U_{-2n} - (6n+7)U_{-2n-1} + 23).$

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $W_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

Corollary 3.27. *For $n \geq 1,$ Narayana-Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kH_{-k} = -(2n+6)H_{-n-1} - (n+3)H_{-n-2} - (n+4)H_{-n-3} + 7.$

(b) $\sum_{k=1}^{n} kH_{-2k} = \frac{1}{9}(-(6n+7)H_{-2n+1} + (3n-1)H_{-2n} - (3n+5)H_{-2n-1} + 13).$

(c) $\sum_{k=1}^{n} kH_{-2k+1} = \frac{1}{9}(-(3n+8)H_{-2n+1} - (3n+5)H_{-2n} - (6n+7)H_{-2n-1} + 29).$

Taking $r = 1, s = 1, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.6. *If $r = 1, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{3}(-(4n+7)W_{-n-1} - (3n+7)W_{-n-2} - 2(n+3)W_{-n-3} + W_2 + W_1)$

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{9}(-(3n+2)W_{-2n+1} + (3n+1)W_{-2n} - 2W_{-2n-1} + W_2 + W_1 - 2W_0)$

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{9}(-W_{-2n+1} - (3n+4)W_{-2n} - 2(3n+2)W_{-2n-1} + 2W_2 - W_1 + 2W_0)$

From the last proposition, we have the following corollary which gives sum formulas of third order Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1, J_2 = 1$).

Corollary 3.28. *For $n \geq 1,$ third order Jacobsthal numbers have the following properties:*

(a) $\sum_{k=1}^{n} kJ_{-k} = \frac{1}{3}(-(4n+7)J_{-n-1} - (3n+7)J_{-n-2} - 2(n+3)J_{-n-3} + 2).$

(b) $\sum_{k=1}^{n} kJ_{-2k} = \frac{1}{9}(-(3n+2)J_{-2n+1} + (3n+1)J_{-2n} - 2J_{-2n-1} + 2).$

(c) $\sum_{k=1}^{n} kJ_{-2k+1} = \frac{1}{9}(-J_{-2n+1} - (3n+4)J_{-2n} - 2(3n+2)J_{-2n-1} + 1).$

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Lucas numbers.

Corollary 3.29. *For $n \geq 1,$ third order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kj_{-k} = \frac{1}{3}(-(4n+7)j_{-n-1} - (3n+7)j_{-n-2} - 2(n+3)j_{-n-3} + 6).$

(b) $\sum_{k=1}^{n} kj_{-2k} = \frac{1}{9}(-(3n+2)j_{-2n+1} + (3n+1)j_{-2n} - 2j_{-2n-1} + 2).$

(c) $\sum_{k=1}^{n} kj_{-2k+1} = \frac{1}{9}(-j_{-2n+1} - (3n+4)j_{-2n} - 2(3n+2)j_{-2n-1} + 13).$

From the last proposition, we have the following corollary which gives sum formulas of modified third order Jacobsthal-Lucas numbers (take $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$).

Corollary 3.30. *For $n \geq 1,$ modified third order Jacobsthal-Lucas numbers have the following properties:*

(a) $\sum_{k=1}^{n} kK_{-k} = \frac{1}{3}(-(4n+7)K_{-n-1} - (3n+7)K_{-n-2} - 2(n+3)K_{-n-3} + 4)$.

(b) $\sum_{k=1}^{n} kK_{-2k} = \frac{1}{9}(-(3n+2)K_{-2n+1} + (3n+1)K_{-2n} - 2K_{-2n-1} - 2)$.

(c) $\sum_{k=1}^{n} kK_{-2k+1} = \frac{1}{9}(-K_{-2n+1} - (3n+4)K_{-2n} - 2(3n+2)K_{-2n-1} + 11)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 0, Q_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal-Perrin numbers.

Corollary 3.31. *For $n \geq 1,$ third order Jacobsthal-Perrin numbers have the following properties:*

(a) $\sum_{k=1}^{n} kQ_{-k} = \frac{1}{3}(-(4n+7)Q_{-n-1} - (3n+7)Q_{-n-2} - 2(n+3)Q_{-n-3} + 2)$.

(b) $\sum_{k=1}^{n} kQ_{-2k} = \frac{1}{9}(-(3n+2)Q_{-2n+1} + (3n+1)Q_{-2n} - 2Q_{-2n-1} - 4)$.

(c) $\sum_{k=1}^{n} kQ_{-2k+1} = \frac{1}{9}(-Q_{-2n+1} - (3n+4)Q_{-2n} - 2(3n+2)Q_{-2n-1} + 10)$.

Taking $r = 2, s = 3, t = 5$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.7. *If $r = 2, s = 3, t = 5$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{81}(-(90n+113)W_{-n-1} - (72n+139)W_{-n-2} - 5(9n+23)W_{-n-3} + 5W_2 + 4W_1 - 20W_0)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{2025}(-(315n+209)W_{-2n+1} + (720n+497)W_{-2n} + 5(90n-11)W_{-2n-1} + 11W_2 + 187W_1 - 530W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{2025}((90n+79)W_{-2n+1} - 11(45n+62)W_{-2n} - 5(315n+209)W_{-2n-1} + 209W_2 - 497W_1 + 55W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of 3-primes numbers (take $W_n = G_n$ with $G_0 = 0, G_1 = 1, G_2 = 2$).

Corollary 3.32. *For $n \geq 1,$ 3-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kG_{-k} = \frac{1}{81}(-(90n+113)G_{-n-1} - (72n+139)G_{-n-2} - 5(9n+23)G_{-n-3} + 14)$.

(b) $\sum_{k=1}^{n} kG_{-2k} = \frac{1}{2025}(-(315n+209)G_{-2n+1} + (720n+497)G_{-2n} + 5(90n-11)G_{-2n-1} + 209)$.

(c) $\sum_{k=1}^{n} kG_{-2k+1} = \frac{1}{2025}((90n+79)G_{-2n+1} - 11(45n+62)G_{-2n} - 5(315n+209)G_{-2n-1} - 79)$.

Taking $W_n = H_n$ with $H_0 = 3, H_1 = 2, H_2 = 10$ in the last proposition, we have the following corollary which presents sum formulas of Lucas 3-primes numbers.

Corollary 3.33. *For $n \geq 1,$ Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kH_{-k} = \frac{1}{81}(-(90n+113)H_{-n-1} - (72n+139)H_{-n-2} - 5(9n+23)H_{-n-3} - 2)$.

(b) $\sum_{k=1}^{n} kH_{-2k} = \frac{1}{2025}(-(315n+209)H_{-2n+1} + (720n+497)H_{-2n} + 5(90n-11)H_{-2n-1} - 1106)$.

(c) $\sum_{k=1}^{n} kH_{-2k+1} = \frac{1}{2025}((90n+79)H_{-2n+1} - 11(45n+62)H_{-2n} - 5(315n+209)H_{-2n-1} + 1261)$.

From the last proposition, we have the following corollary which gives sum formulas of modified 3-primes numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 3.34. *For $n \geq 1,$ modified 3-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kE_{-k} = \frac{1}{81}(-(90n+113)E_{-n-1} - (72n+139)E_{-n-2} - 5(9n+23)E_{-n-3} + 9)$.

(b) $\sum_{k=1}^{n} kE_{-2k} = \frac{1}{2025}(-(315n+209)E_{-2n+1} + (720n+497)E_{-2n} + 5(90n-11)E_{-2n-1} + 198)$.

(c) $\sum_{k=1}^{n} kE_{-2k+1} = \frac{1}{2025}((90n+79)E_{-2n+1} - 11(45n+62)E_{-2n} - 5(315n+209)E_{-2n-1} - 288)$.

Taking $r = 5, s = 3, t = 2$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.8. *If $r = 5, s = 3, t = 2$ then for $n \geq 1$ we have the following formulas:*

(a) $\sum_{k=1}^{n} kW_{-k} = \frac{1}{81}(-(90n + 107) W_{-n-1} - (45n + 67) W_{-n-2} - 2(9n + 17) W_{-n-3} - W_2 + 13W_1 - 20W_0)$.

(b) $\sum_{k=1}^{n} kW_{-2k} = \frac{1}{405}(-(63n + 10) W_{-2n+1} + (333n + 49) W_{-2n} + 2(18n - 19) W_{-2n-1} + 19W_2 - 85W_1 - 106W_0)$.

(c) $\sum_{k=1}^{n} kW_{-2k+1} = \frac{1}{405}((18n - 1) W_{-2n+1} - 17(9n + 4) W_{-2n} - 2(63n + 10) W_{-2n-1} + 10W_2 - 49W_1 + 38W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse 3-primes numbers (take $W_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 5$).

Corollary 3.35. *For $n \geq 1$, reverse 3-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kN_{-k} = \frac{1}{81}(-(90n + 107) N_{-n-1} - (45n + 67) N_{-n-2} - 2(9n + 17) N_{-n-3} + 8)$.

(b) $\sum_{k=1}^{n} kN_{-2k} = \frac{1}{405}(-(63n + 10) N_{-2n+1} + (333n + 49) N_{-2n} + 2(18n - 19) N_{-2n-1} + 10)$.

(c) $\sum_{k=1}^{n} kN_{-2k+1} = \frac{1}{405}((18n - 1) N_{-2n+1} - 17(9n + 4) N_{-2n} - 2(63n + 10) N_{-2n-1} + 1)$.

Taking $W_n = S_n$ with $S_0 = 3, S_1 = 5, S_2 = 31$ in the last proposition, we have the following corollary which presents sum formulas of reverse Lucas 3-primes numbers.

Corollary 3.36. *For $n \geq 1$, reverse Lucas 3-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kS_{-k} = \frac{1}{81}(-(90n + 107) S_{-n-1} - (45n + 67) S_{-n-2} - 2(9n + 17) S_{-n-3} - 26)$.

(b) $\sum_{k=1}^{n} kS_{-2k} = \frac{1}{405}(-(63n + 10) S_{-2n+1} + (333n + 49) S_{-2n} + 2(18n - 19) S_{-2n-1} - 154)$.

(c) $\sum_{k=1}^{n} kS_{-2k+1} = \frac{1}{405}((18n - 1) S_{-2n+1} - 17(9n + 4) S_{-2n} - 2(63n + 10) S_{-2n-1} + 179)$.

From the last proposition, we have the following corollary which gives sum formulas of reverse modified 3-primes numbers (take $W_n = U_n$ with $U_0 = 0, U_1 = 1, U_2 = 4$).

Corollary 3.37. *For $n \geq 1$, reverse modified 3-primes numbers have the following properties:*

(a) $\sum_{k=1}^{n} kU_{-k} = \frac{1}{81}(-(90n + 107) U_{-n-1} - (45n + 67) U_{-n-2} - 2(9n + 17) U_{-n-3} + 9)$.

(b) $\sum_{k=1}^{n} kU_{-2k} = \frac{1}{405}(-(63n + 10) U_{-2n+1} + (333n + 49) U_{-2n} + 2(18n - 19) U_{-2n-1} - 9)$.

(c) $\sum_{k=1}^{n} kU_{-2k+1} = \frac{1}{405}((18n - 1) U_{-2n+1} - 17(9n + 4) U_{-2n} - 2(63n + 10) U_{-2n-1} - 9)$.

4 CONCLUSION

In this work, a number of sum identities were discovered and proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Tribonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the special cases of the generalized Tribonacci sequences. All the listed identities may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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