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A Moments of Generalized Two Variable Szasz Operators

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

In the present paper some moments of generalized one-dimensional Szasz operator in the spaces of continuously differentiable functions is found. Also, in two-dimensional case a generalization of Szasz operator is considered and certain moments of this operator in the spaces of continuously differentiable functions is found.

Keywords: Szasz operator; generalized Szasz operator; spaces of continuously differentiable functions; moments.

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1. INTRODUCTION

This work is devoted to the computing of some moments of generalized Szasz operator. To reveal what novelties this paper brings, we briefly present both a background and some historical comments. The main problem of approximation theory consists in finding for a complicated function a close-by simple function. At 70 years old, Weierstrass (1815–1897) proved the density

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of the algebraic polynomials in the space $C[a,b]$ and of the trigonometric polynomial in $\tilde{C}[a,b]$. Weierstrass approximation theorem stating that every continuous function on a bounded interval can be approximated to arbitrary accuracy by polynomials is such an important example for this process and has been played the significant role in the development of analysis. By using probability theory Bernstein [1] proved the Weierstrass theorem and defined approximate polynomials known as Bernstein polynomials in the literature (see [2]). Namely, in 1912, Bernstein [1] constructed an approximate polynomials in the form

$$B_n(f;x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}(x)^k(1-x)^{n-k}, \quad 0\le x\le 1,$$

where $f\in C[0,1]$. In [3,4] Stancu introduced some generalizations of Bernstein polynomials.

In 1950, Szasz defined and studied the approximation properties of the following operators

$$S_n(f;x)=e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}, \quad 0\le x<\infty, \tag{1.1}$$

where $f\in C[0,\infty)$ satisfies exponential-type growth condition [5]. In 1962, Schurer [6] introduced and studied the approximation properties of linear positive operators. An extension in q-calculus of Szasz operators was constructed by Aral [7] who formulated also a Voronovskaya theorem related to q-derivatives for these operators. After that several other researchers have studied in this direction and obtained different approximation properties of many operators [8,9,2]. The weighted Korovkin-type theorems were proved by Gadzhiev [10]. Recently, approximation theorems for generalized Szasz operators and Bernstein-Chlodowsky polynomials was proved in [11,12].

In this paper certain moments of generalized one-dimensional and two-dimensional Szasz operator are found.

2. PRELIMINARIES

Now we give the definitions of one-dimensional and two-dimensional generalized Szasz operator.

Definition 2.1. Let

$$S_{n,r}(f;x)=e^{-nx}\sum_{k=0}^{\infty}\sum_{i=0}^{r}\frac{f^{(i)}\left(\frac{k}{n}\right)}{i!}\left(x-\frac{k}{n}\right)^i\frac{(nx)^k}{k!},$$
$$f\in C^r[0,\infty). \tag{2.1}$$

The operator $S_{n,r}(f;x)$ defined by (2.1) is called the generalized one-dimensional Szasz operator.

Remark 2.1. Note that for $r=0$ the operator $S_{n,r}(f;x)$ is coincide with classical Szasz operator (1.1).

Suppose that $C^{(\ell,m)}(R_+^2)=\left\{f:\ \frac{\partial^{i+j}f}{\partial x^i\partial y^j}\in C(R_+^2),\ 0\le i\le \ell,\ 0\le j\le m\right\}$, where $C(R_+^2)$ the spaces of continuously functions on R_+^2.

Definition 2.2. Let

$$S_{n,m,r}(f;x,y)=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{r}\frac{1}{i!}\left[\left(x-\frac{k}{n}\right)\frac{\partial}{\partial x}+\left(y-\frac{l}{m}\right)\frac{\partial}{\partial y}\right]^i f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}, \tag{2.2}$$

where $f\in C^{(\ell,m)}(R_+^2),\ 0\le \ell,\ m\le r,\ \ l,m=0,1,\dots,r$ and we denote

$$\left[\left(x-\frac{k}{n}\right)\frac{\partial}{\partial x}+\left(y-\frac{l}{m}\right)\frac{\partial}{\partial y}\right]^i f=\sum_{p=0}^{i}C_i^p\left(x-\frac{k}{n}\right)^{i-p}\left(y-\frac{l}{m}\right)^p\frac{\partial^i f}{\partial^{i-p}x\partial^p y}.$$

The operator $S_{n,m,r}(f;x,y)$ defined by (2.2) is called the generalized two-dimensional Szasz operator.

Remark 2.2. Note that for $r=0$ the operator $S_{n,r}(f;x)$ is coincide with classical two-dimensional Szasz operator. In other words

$$S_{n,m}(f;x,y)=e^{-nx}e^{-my}\sum_{k,l=0}^{\infty} f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}.$$

3. MAIN RESULTS

Now we reduce the main results of this paper.

Theorem 3.1. For the generalized Szasz operator the following equalities hold:

$$S_{n,r}(1;x)=1,\quad r\geq 0,$$

$$S_{n,r}(t;x)=x,\quad r\geq 0,$$

$$S_{n,r}(t^2;x)=\begin{cases}\dfrac{x}{n}+x^2, & r=0\\ x^2-\dfrac{x}{n}, & r=1\\ x^2, & r\geq 2.\end{cases}$$

Proof. Let us take $f(t)=1$ in (2.1). We get

$$S_{n,r}(f(t);x)=S_{n,r}(1;x)=$$

$$e^{-nx}\sum_{k=0}^{\infty}\sum_{i=0}^{r}\frac{f^{(i)}\left(\frac{k}{n}\right)}{i!}\left(x-\frac{k}{n}\right)^i\frac{(nx)^k}{k!}=1.$$

$$f^{(0)}=f\equiv 1,$$

$$f^{(i)}\left(\frac{k}{n}\right)=0,\qquad i\geq 1.$$

Suppose $f(t)=t$ and let us take $t=\dfrac{k}{n}$. We have

$$S_{n,r}(f(t);x)=S_{n,r}(t;x)=$$

$$e^{-nx}\sum_{k=0}^{\infty}\sum_{i=0}^{r}\frac{f^{(i)}\left(\frac{k}{n}\right)}{i!}\left(x-\frac{k}{n}\right)^i\frac{(nx)^k}{k!}=x,$$

Where

$$f^{(0)}=f\left(\frac{k}{n}\right)\equiv\frac{k}{n},$$

$$f^{(1)}\left(\frac{k}{n}\right)=1,$$

and

$$f^{(i)}\left(\frac{k}{n}\right)=0,\qquad i\geq 2.$$

Therefore

$$S_{n,r}(t;x)=e^{-nx}\sum_{k=0}^{\infty}\left(\frac{k}{n}+\left(x-\frac{k}{n}\right)\right)\frac{(nx)^k}{k!}=x.$$

Finally we take $f(t)=t^2$. We get

$$S_{n,r}(f(t);x)=S_{n,r}(t^2;x)=$$

$$e^{-nx}\sum_{k=0}^{\infty}\sum_{i=0}^{r}\frac{f^{(i)}\left(\frac{k}{n}\right)}{i!}\left(x-\frac{k}{n}\right)^i\frac{(nx)^k}{k!}=x^2,$$

Where

$$f^{(0)}=f\left(\frac{k}{n}\right)\equiv\left(\frac{k}{n}\right)^2,$$

$$f^{(1)}\left(\frac{k}{n}\right)=2\frac{k}{n},$$

$$f^{(2)}\left(\frac{k}{n}\right)=2,$$

and

$$f^{(i)}\left(\frac{k}{n}\right)=0,\qquad i\geq 3.$$

It is obvious that for $r=0$ we have

$$S_{n,0}(t^2;x)=e^{-nx}\sum_{k=0}^{\infty}\frac{k^2}{n^2}\frac{(nx)^k}{k!}=\frac{x}{n}+x^2.$$

For $r=1$ we get

$$S_{n,1}(t^2;x)=e^{-nx}\sum_{k=0}^{\infty}\left(\frac{k^2}{n^2}+2\frac{k}{n}\left(x-\frac{k}{n}\right)\right)\frac{(nx)^k}{k!}=x^2-\frac{x}{n}.$$

Finally, for $r\geq 2$ we have that

$$S_{n,r}(t^2;x)=e^{-nx}\sum_{k=0}^{\infty}\left(\frac{k^2}{n^2}+2\frac{k}{n}\left(x-\frac{k}{n}\right)+\left(x-\frac{k}{n}\right)^2\right)\frac{(nx)^k}{k!}=x^2.$$

This completes the proof of the Theorem 3.1.

Remark 3.1. Note that for Theorem 3.1 in different form was proved in [13].

Theorem 3.2. Let $f\in C^{(i,j)}\left(R_+^2\right),\, 0\leq i,\, j\leq r.$ Then the following equalities hold:

$$S_{n,m,r}(1;x,y)=1$$
$$S_{n,m,r}(t;x,y)=x,$$
$$S_{n,m,r}(\tau;x,y)=y,$$
$$S_{n,m,r}(t^2;x,y)=x^2,$$
$$S_{n,m,r}(\tau^2;x,y)=y^2,$$
$$S_{n,m,r}(t\tau;x,y)=xy.$$

Proof: It is obvious that, if $f(t,\tau)\equiv 1$, then we have

$$S_{n,m,r}(1;x,y)=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{r}\frac{1}{i!}\left[\left(x-\frac{k}{n}\right)\frac{\partial}{\partial x}+\left(y-\frac{l}{m}\right)\frac{\partial}{\partial y}\right]^i f\left(\frac{k}{n},\frac{l}{m}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=1.$$

Obviously,

$$f^{(0,0)}=f\equiv 1$$

and

$$f^{(i,j)}\left(\frac{k}{n}\right)=0,\qquad i+j\geq 1.$$

Let us take $f(t,\tau)\equiv t.$ Then we get

$$S_{n,m,r}(t;x,y)=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{r}\frac{1}{i!}\left[\left(x-\frac{k}{n}\right)\frac{\partial}{\partial x}+\left(y-\frac{l}{m}\right)\frac{\partial}{\partial y}\right]^i\frac{k}{n}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=$$

$$=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{1}\frac{1}{i!}\left[\left(x-\frac{k}{n}\right)\frac{\partial}{\partial x}+\left(y-\frac{l}{m}\right)\frac{\partial}{\partial y}\right]^i\frac{k}{n}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=$$

$$=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\frac{k}{n}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}+e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(x-\frac{k}{n}\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=x,$$

where

$$f^{(0,0)}=f\left(\frac{k}{n}\right)\equiv\frac{k}{n},$$

$$f^{(1,0)}\left(\frac{k}{n}\right)=1,\; f^{(0,1)}\left(\frac{k}{n}\right)=0$$

and

$$f^{(i,j)}\left(\frac{k}{n}\right)=0,\qquad i+j\geq 2.$$

In the same way, if $f(t,\tau)=\tau$ then we have

$$S_{n,m,r}(\tau;x,y)=\frac{l}{m}+y-\frac{l}{m}=y, \qquad \tau=\frac{l}{m}.$$

Obviously,

$$f^{(0,0)}=f\left(\frac{l}{m}\right)\equiv\frac{l}{m},$$

$$f^{(1,0)}\left(\frac{l}{m}\right)=0, \quad f^{(0,1)}\left(\frac{l}{m}\right)=1,$$

$$f^{(i,j)}(\tau)=0, \qquad i+j\geq 2.$$

Further, we get

$$S_{n,m,r}(f(t,\tau);x,y)=S_{n,m,r}(t^2;x,y)=$$

$$=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{2}\frac{1}{i!}\left[\left(x-\frac{k}{n}\right)\frac{\partial}{\partial x}+\left(y-\frac{l}{m}\right)\frac{\partial}{\partial y}\right]^i\frac{k^2}{n^2}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=$$

$$=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{k^2}{n^2}+2\frac{k}{n}\left(x-\frac{k}{n}\right)+\left(x-\frac{k}{n}\right)^2\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=$$

$$=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{k}{n}+\left(x-\frac{k}{n}\right)\right)^2\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=x^2,$$

where

$$f^{(0,0)}=f\left(\frac{k}{n}\right)\equiv\left(\frac{k}{n}\right)^2,$$

$$f^{(1,0)}\left(\frac{k}{n}\right)=2\frac{k}{n}, \; f^{(1,0)}\left(\frac{l}{m}\right)=0,$$

$$f^{(2,0)}\left(\frac{k}{n}\right)=2, \quad f^{(0,2)}\left(\frac{k}{n}\right)=0$$

and

$$f^{(i,j)}\left(\frac{k}{n}\right)=0, \qquad i+j\geq 3.$$

In the same way, if $f(t,\tau)=\tau^2$ then we have

$$S_{n,m,r}(\tau^2;x,y)=y^2$$

After some calculation, we get

$$S_{n,m,r}(f(t,\tau);x,y)=S_{n,m,r}(t\tau;x,y)=$$

$$=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{i=0}^{2}\frac{1}{i!}\left[\left(x-\frac{k}{n}\right)\frac{\partial}{\partial x}+\left(y-\frac{l}{m}\right)\frac{\partial}{\partial y}\right]^i\frac{k}{n}\frac{l}{m}\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=$$

$$=e^{-nx-my}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\left(\frac{k}{n}\frac{l}{m}+\frac{l}{m}\left(x-\frac{k}{n}\right)+\frac{k}{n}\left(y-\frac{l}{m}\right)+\left(x-\frac{k}{n}\right)\left(y-\frac{l}{m}\right)\right)\frac{(nx)^k}{k!}\frac{(my)^l}{l!}=xy.$$

and this ends the proof of the Theorem 3.2.

4. CONCLUSION

Thus, some moments of generalized one-dimensional Szasz operator in the spaces of continuously differentiable functions is found. Also, in two-dimensional case a generalization of Szasz operator is considered and certain moments of this operator in the spaces of continuously differentiable functions is found.

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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