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Amply Essential Supplemented Modules

Celil Nebiyev[1], Hasan Hüseyin Ökten[2*] and Ayten Pekin[3]

[1] *Department of Mathematics, Ondokuz Mayıs University, Kurupelit-Atakum, Samsun, Turkey.*
[2] *Technical Sciences Vocational School, Amasya University, Amasya, Turkey.*
[3] *Department of Mathematics, Istanbul University, Vezneciler, Istanbul, Turkey.*

Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

In this work, amply essential supplemented modules are defined and some properties of these modules are investigated. We prove that every $\pi-$projective and essential supplemented module is amply essential supplemented. We also prove that every factor module and every homomorphic image of an amply essential supplemented module are amply essential supplemented. Let M be a projective and essential supplemented $R-$module. Then every finitely $M-$generated $R-$module is amply essential supplemented.

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**Corresponding author: E-mail: hokten@gmail.com*

1 INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an $R-$module. We will denote a submodule N of M by $N \leq M$. Let M be an $R-$module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a small submodule of M and denoted by $N \ll M$. A submodule N of an R-module M is called an essential submodule of M and denoted by $N \trianglelefteq M$ in case $K \cap N \neq 0$ for every submodule $K \neq 0$, or equvalently, $N \cap L = 0$ for $L \leq M$ implies that $L = 0$. Let M be an $R-$module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a supplement of U in M. M is called a supplemented module if every submodule of M has a supplement in M. Let M be an $R-$module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement $V^{'}$ with $V^{'} \leq V$, we say U has ample supplements in M. If every submodule of M has ample supplements in M, then M is called an amply supplemented module. The intersection of maximal submodules of an R-module M is called the radical of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an $R-$module. It is defined β^* relation on the submodules of M by $A\beta^*B$ with $A \leq M$ and $B \leq M$ if and only if for each $T \leq M$ such that $A+T = M$ then $B+T = M$ and for each $K \leq M$ such that $B + K = M$ then $A + K = M$.

More details about supplemented modules are in [1], [2] and [3]. The definition of β^* relation and some properties of this relation are in [4].

Definition 1.1. See [5] Let M be an $R-$module. If every essential submodule of M has a supplement in M, then M is called an essential supplemented (or briefly, e-supplemented) module.

Definition 1.2. See [5] Let M be an $R-$module and $X \leq M$. If X is a supplement of an essential submodule of M, then X is called an e-supplement submodule in M.

Lemma 1.1. *See [5] Let M be an essential supplemented module. Then every finitely $M-$generated $R-$module is essential supplemented.*

2 AMPLY ESSENTIAL SUPPLEMENTED MODULES

Definition 2.1. Let M be an $R-$module. If every essential submodule of M has ample supplements in M, then M is called an amply essential supplemented (or briefly, amply e-supplemented) module.

Clearly, every amply essential supplemented module is essential supplemented.

Proposition 2.1. *Let M be an amply essential supplemented module. Then $M/RadM$ have no proper essential submodules.*

Proof. Since M is amply essential supplemented, then M is essential supplemented. Then by ([5], Proposition 2.5), $M/RadM$ have no proper essential submodules. □

Lemma 2.1. *Let M be an amply essential supplemented module. Then every factor module of M is amply essential supplemented.*

Proof. Let $\frac{M}{K}$ be any factor module of M. Let $\frac{U}{K} \trianglelefteq \frac{M}{K}$ and $\frac{M}{K} = \frac{U}{K} + \frac{V}{K}$. Then $U \trianglelefteq M$ and $M = U + V$. Since M is amply eessential supplemented, U has a supplement X in M with $X \leq V$. Since $K \leq U$, by [2], 41.1 (7), $\frac{X+K}{K}$ is a supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Moreover, $\frac{X+K}{K} \leq \frac{V}{K}$. Hence $\frac{M}{K}$ is amply essential supplemented. □

Corollary 2.2. *Every homomorphic image of an amply essential supplemented module is amply essential supplemented.*

Proof. Clear from Lemma 2.1. □

Definition 2.2. Let M be an $R-$module. If every proper essential submodule of M is small in M or M have not proper essential submodules, then M is called an e-hollow module.

Lemma 2.3. *Every e-hollow module is amply essential supplemented.*

Proof. Clear from definitions. □

Corollary 2.4. *Every e-hollow module is essential supplemented.*

Proof. Clear from Lemma 2.3, since every amply essential supplemented module is essential supple-mented. □

Lemma 2.5. *If M is a π-projective and essential supplemented module, then M is an amply essential supplemented module.*

Proof. Let $U \trianglelefteq M$, $M = U + V$ and X be a supplement of U in M. Since M is π -projective and $M = U + V$, there exists an R -module homomorphism $f : M \to M$ such that $Imf \subset V$ and $Im(1-f) \subset U$. So, we have $M = f(M) + (1-f)(M) = f(U) + f(X) + U = U + f(X)$. Suppose that $a \in U \cap f(X)$. Since $a \in f(X)$, then there exists $x \in X$ such that $a = f(x)$. Since $a = f(x) = f(x) - x + x = x - (1-f)(x)$ and $(1-f)(x) \in U$, we have $x = a + (1-f)(x) \in U$. Thus $x \in U \cap X$ and so, $a = f(x) \in f(U \cap X)$. Therefore we have $U \cap f(X) \leq f(U \cap X) \ll f(X)$. This means $f(X)$ is a supplement of U in M. Moreover, $f(X) \subset V$. Therefore M is amply essential supplemented. □

Corollary 2.6. *If M is a projective and essential supplemented module, then M is an amply essential supplemented module.*

Proof. Clear from Lemma 2.5. □

Lemma 2.7. *Let M be a π-projective $R-$module. If every essential submodule of M is β^* equivalent to an e-supplement submodule in M, then M is amply essential supplemented.*

Proof. By [5], Lemma 2.13, M is essential supplemented. Then by Lemma 2.5, M is amply essential supplemented. □

Corollary 2.8. *Let M be a projective $R-$module. If every essential submodule of M is β^* equivalent to an e-supplement submodule in M, then M is amply essential supplemented.*

Proof. Clear from Lemma 2.7. □

Lemma 2.9. *Let Λ be a finite index set and $\{M_\lambda\}_\Lambda$ be a family of projective $R-$modules. If M_λ is essential supplemented for every $\lambda \in \Lambda$, then $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply essential supplemented.*

Proof. Since M_λ is essential supplemented for every $\lambda \in \Lambda$, by ([5], Corollary 2.8), $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is essential supplemented. Since M_λ is projective for every $\lambda \in \Lambda$, by ([2], 18.1), $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective. Since $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is projective and essential supplemented, by Corollary 2.6, $\underset{\lambda\in\Lambda}{\oplus} M_\lambda$ is amply essential supplemented. □

Corollary 2.10. *Let M be a projective $R-$module. If M is essential supplemented, then $M^{(\Lambda)}$ is amply essential supplemented for every finite index set Λ.*

Proof. Clear from Lemma 2.9. □

Corollary 2.11. *Let M be a projective $R-$module. If M is essential supplemented, then every finitely $M-$generated $R-$module is amply essential supplemented.*

Proof. Let N be a finitely $M-$generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : M^{(\Lambda)} \longrightarrow N$. Since M is projective and essential supplemented, by Lemma 2.9, $M^{(\Lambda)}$ is amply essential supplemented. Then by Corollary 2.2, N is amply essential supplemented. □

Lemma 2.12. *Let M be an $R-$module. If every submodule of M is essential supplemented, then M is amply essential supplemented.*

Proof. Let $U \trianglelefteq M$ and $M = U + V$ with $V \leq M$. Since $U \trianglelefteq M$, $U \cap V \trianglelefteq V$. By hypothesis, V is essential supplemented. Then $U \cap V$ has a supplement X in V. By this, $V = U \cap V + X$ and $U \cap X = U \cap V \cap X \ll X$. Then $M = U + V = U + U \cap V + X = U + X$ and $U \cap X \ll X$. Hence M is amply essential supplemented. □

Lemma 2.13. *Let R be any ring. Then every $R-$module is essential supplemented if and only if every $R-$module is amply essential supplemented.*

Proof. ($\Longrightarrow$) Let M be any $R-$module. Since every $R-$module is essential supplemented, every submodule of M is essential supplemented. Then by Lemma 2.12, M is amply essential supplemented.

($\Longleftarrow$) Clear. □

Proposition 2.2. *Let R be a ring. The following assertions are equivalent.*

$(i)_R R$ *is essential supplemented.*

$(ii)_R R$ *is amply essential supplemented.*

(iii) *Every finitely generated $R-$module is essential supplemented.*

(iv) *Every finitely generated $R-$module is amply essential supplemented.*

Proof. $(i) \Longleftrightarrow (ii)$ Clear from Corollary 2.6, since $_RR$ is projective.

$(i) \Longrightarrow (iii)$ Clear from Lemma 1.1.

$(iii) \Longrightarrow (iv)$ Let M be a finitely generated $R-$module. Then there exist a finite index set Λ and an $R-$module epimorphism $f : R^{(\Lambda)} \longrightarrow M$. Since every finitely generated $R-$module is essential supplemented, $R^{(\Lambda)}$ is essential supplemented. Since $_RR$ is projective, by ([2], 18.1), $R^{(\Lambda)}$ is also projective. Then by Corollary 2.6, $R^{(\Lambda)}$ is amply essential supplemented. Since $f : R^{(\Lambda)} \longrightarrow M$ is an $R-$module epimorphism, by Corollary 2.2, M is also amply essential supplemented.

$(iv) \Longrightarrow (i)$ Clear. □

3 CONCLUSION

Let R be a ring. Then $_RR$ is amply essential supplemented if and only if every finitely generated $R-$module is amply essential supplemented.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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